

THE ASTROPHYSICAL JOURNAL

AN INTERNATIONAL REVIEW OF SPECTROSCOPY
AND ASTRONOMICAL PHYSICS

VOLUME IX

FEBRUARY 1899

NUMBER 2

THE ORBIT OF η AQUILAE.

By W. H. WRIGHT.

THE variable velocity in the line of sight of the star η Aquilae was announced by Dr. A. B  lopolsky in 1895, and discussed by him in some detail in this JOURNAL.¹ This star is on the regular observing list made out some years ago by Professor Campbell for radial velocity determinations with the Mills' spectrograph. During the past summer its spectrum has been systematically photographed. In all, twenty-seven spectrograms have been secured by Professor Campbell and myself, and he has asked me, with the consent of the Director, to discuss the results.

The methods employed in the processes of photography, measurement, and reduction are those described by Professor Campbell in the October number of this JOURNAL. The matter of the selection of suitable lines for measurement requires the exercise of some discrimination on account of the nature of the spectrum, which is intermediate in type between I and II.² The

¹ 6, 393-399, 1897.

² There seems to exist quite a range of opinion regarding the type of the spectrum of η Aquilae. One writer describes it as being of solar type. Another places it between II and III, and mentions its remarkable resemblance to that of δ Cephei, which is regarded by him as being of the same type as Arcturus. There is undoubtedly a resemblance between the spectra of η Aquilae and δ Cephei, the lines of the latter being, however, better defined, but it is difficult to reconcile their appearance with the

lines, which are fairly numerous, have the general characteristics of breadth and haziness, which tend to make them objectionable for purposes of accurate measurement. The difficulty, which is greatly augmented when the negative is at all underexposed, arises, not so much from an inability of the observer to determine the center of a broad line, as from the vitiating effect of companion lines which under more propitious circumstances would be resolved. The question of interference of companion lines has been discussed by Professor Campbell in the article referred to above, and is only mentioned in the present instance to emphasize its importance in spectra of the type of η Aquilae. In the case of stars of type II the presence of unresolved companions may sometimes be inferred from solar spectrum analogies. In other cases, such as the present one, recourse must be had to the appearance of the line, and, in a sense, to the general agreement of its position (as measured on many plates) with that of a large number of other lines. To illustrate, the lines $\lambda\lambda$ 4344.670 and 4359.784 are broad, and their centers are shifted to the violet by an amount greater than the uncertainty of measurement, and they have not been employed. The following is a list of lines that have been used in determining the velocities of this star:

λ	λ	λ	λ
4278.390	4330.405	4341.530	4415.722
82.565	30.866	52.908	16.985
94.936	31.811	55.257	17.884
4313.034	33.925	76.107	30.785
16.962	37.216	88.571	35.851
25.152	38.084	89.413	37.112
25.939	38.430	94.225	42.510
28.080	40.634 (H_γ)	99.935	

On an average about twelve lines were measured on each plate, the available number depending upon the quality of the negative.

statement that they resemble the spectrum of Arcturus. While they have many lines in common with the latter, their general appearance is entirely different. The Draper Catalogue classes the star as "G 2," which, barring the interrogation, would place it between I and II, where, in the opinion of the writer, it belongs.

It is hardly necessary to detail here the results obtained from the individual lines of each plate. To illustrate their general degree of accordance, however, the results from plates 791 B, 901 A, and 994 A are given in full:

		791 B	901 A	994 A	
	500	4294.936	-35.5		
	750	4303.337	34.4		
Σ	- 6007	13.034	31.4	+27.9	+28.8
	500	16.962	32.3	30.0	
	750	25.152	35.1	25.9	30.5
	500	25.939	33.2	26.7	30.3
	500	28.080	36.3		32.7
	600	30.866	31.9		
	1100	37.216	33.2	26.4	30.6
Σ	- 410	38.080	34.8	26.2	
		38.430	38.9		
	HY	40.634	26.3		
		41.530	35.8		
	3500	55.257	37.1		29.0
	500	69.941	32.2		
	500	76.107	35.0	24.7	26.1
	500	94.225	34.8	21.3	26.9
Σ	- 800	99.935	35.1		24.3
	500	4415.722	32.6		
	500	16.985	21.7		
	450	30.785	34.4		
	600	35.851	23.9		
	300	37.111	36.1		
	300	42.510	33.2	27.0	29.7
Means	- - -	-34.44	+25.66	+28.89	
Correction for curvature	- - -	- 0.76	- 0.63	- 0.69	
Reduction to \odot	- - -	+ 9.98	-16.98	-27.38	
		-25.2	+ 8.0	+ 0.8	

Results of the measurements are given in column 5 of Table I, which will not need explanation except with regard to the system of weights used. The assignment of weights is generally more or less of an arbitrary matter, and after some consideration the following system was adopted:

Plates upon which less than eight lines were measured, $\text{wt.} = \frac{1}{3}$.

Plates upon which between eight and twelve lines were measured, $\text{wt.} = \frac{2}{3}$.

Plates upon which more than twelve lines were measured, $\text{wt.} = 1$.

Assuming with B  lopol'sky an orbital period equal to that of the light variation of the star (7.176 days), the velocity curve

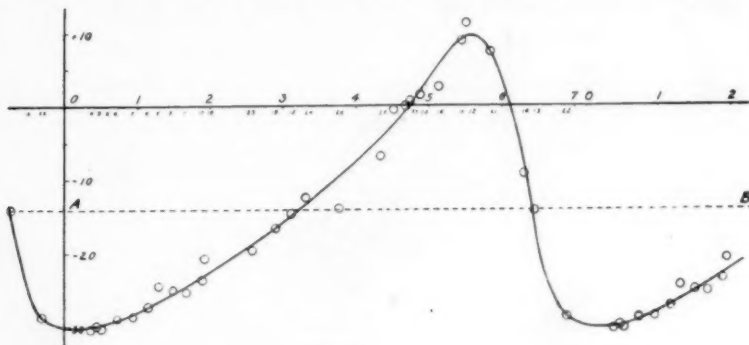


FIG. 1.

was drawn in the usual manner. In Fig. 1 the abscissae represent time intervals counted from the preceding light maxima,¹ and the ordinates, velocities in the line of sight in kilometers per second. The dotted line *AB* indicates the velocity of the center of mass of the system, the upper and lower areas having been adjusted to equality by means of a planimeter. Adopting the formul  e and notation of Lehmann-Filh  s,² the following constants and elements have been computed:

ELEMENTS I.

$A = 23.9$ km, greatest positive velocity³ in the line of sight.

$B = 16.2$ km, numerical value of greatest negative velocity³ in line of sight.

$t_2 - t_1 = 3.95^d$, time during which the velocity curve is below the line *AB*.

¹ In computing the light ephemeris Schur's elements have been used: *A. N.*, 3282; also Chandler "Third Catalogue of Variable Stars," *A. J.*, 379.

² *A. N.*, 3242.

³ Referred to the center of mass of the system.

$\frac{z_1}{z_2} = -\frac{74}{188}$, ratio of greatest distances of star above and below reference plane.

$u_1 = 101^\circ$, point for which velocity in line of sight = 0.

$u_2 = 259^\circ$, point for which velocity in line of sight = 0.

$U = 7.176^d$, period (assumed).

$\omega = 65.79^\circ$, position of periastron.

$e = 0.47$, eccentricity of orbit.

$T = 6.176$, time of periastron passage.

$a \sin i = 1,545,000$ km.

The residuals for these elements are given in Table I, column 7. It will be seen that the one corresponding to observation

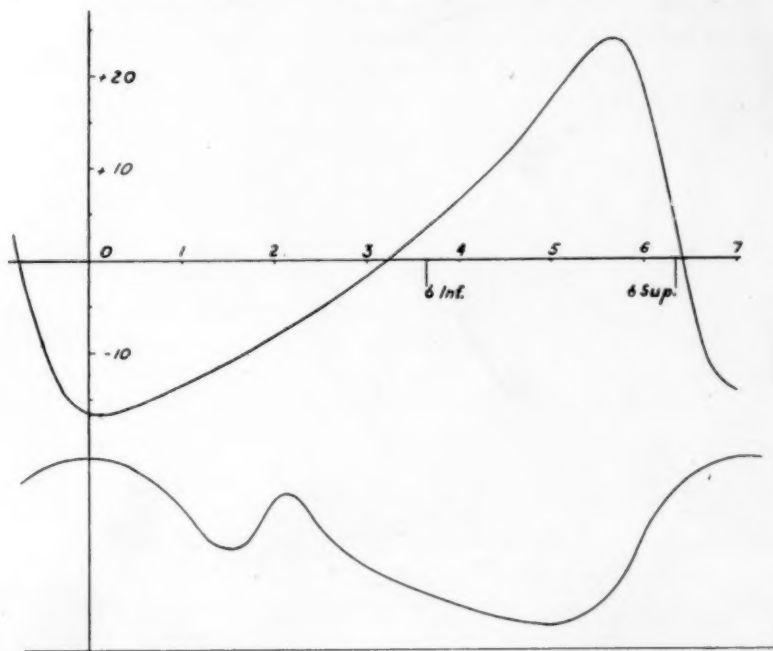


FIG. 2.

No. 26 is larger than would be expected from the general accordance of the rest. A somewhat extended experience at this Observatory in the determinations of radial velocities of stars

leads us to the conclusion that a residual of 4.8 km is not to be expected as the result of ordinary accidental errors of observation. It is above the limit set by the various criteria for the rejection of doubtful observations, and was taken to indicate the presence during some part of the manipulation of the plate of an abnormal source of error. The observation was accordingly rejected.

The velocity curve computed from *Elements I* is given in the upper part of Fig. 2.

A final adjustment of the observations was then undertaken, and ten normal residuals formed, as indicated in Table II. The form of the equations of observation is that developed by Lehmann-Filhés, except that the correction to the period has been assumed to be zero, and the correction to the velocity of the system introduced as an unknown. The following observation equations result:

wt.	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>m</i>
2, +	10.0δ <i>v</i>	+ 15.2δ <i>e</i>	- 4.8δ <i>ω</i>	- 2.6δ <i>T</i>	- 7.9δ <i>K</i>	+ 3.6 = 0
2, "	"	+ 20.3	+ 0.8	- 4.7	- 6.9	+ 4.3 = 0
2, "	"	+ 19.2	+ 4.8	- 5.6	- 5.5	- 1.3 = 0
3, "	"	+ 13.7	+ 8.1	- 6.1	- 3.6	- 1.7 = 0
3, "	"	+ 1.1	+ 10.9	- 7.2	- 0.4	- 4.2 = 0
2, "	"	- 21.4	+ 10.0	- 10.8	+ 5.7	- 2.7 = 0
2, "	"	- 23.8	+ 7.1	- 12.3	+ 8.1	+ 5.0 = 0
2, "	"	+ 6.4	- 7.9	- 1.3	+ 11.9	- 2.7 = 0
1, "	"	- 20.6	- 28.6	+ 51.2	+ 1.3	+ 3.0 = 0
1, "	"	- 17.7	- 18.6	+ 13.8	- 6.8	+ 12.0 = 0

In order to render these equations more nearly homogeneous, the coefficients *a*, *e*, and the residuals *m* have been multiplied by ten. Following are the resulting normal equations, the logarithms of the coefficients being given instead of their numerical values:

3 .301030δ <i>v</i>	+ 2 .578639δ <i>e</i>	+ 2 .474216ωδ
2 .578639	3 .737272	2 .687529
2 .474216	2 .687529	3 .349472
2 _N 694605	3 _N 011993	3 _N 399847
1 _N 826075	3 _N 102091	1 .623249

$$\begin{array}{rcl}
 + 2_{\text{n}} 694605 \delta T + 1_{\text{n}} 826075 \delta K + 1.986772 = 0 \\
 3_{\text{n}} 011993 & 3_{\text{n}} 102091 & 2_{\text{n}} 448706 = 0 \\
 3_{\text{n}} 399847 & 1.623249 & 2_{\text{n}} 669317 = 0 \\
 3.572639 & 2_{\text{n}} 136721 & 2_{\text{n}} 531479 = 0 \\
 2_{\text{n}} 136721 & 2.929930 & 2_{\text{n}} 232996 = 0
 \end{array}$$

The solution of these gives:

$$\begin{array}{lll}
 \delta v = -0.065 \text{ km} & \pm 0.06 & \pm 0.17 \\
 \delta e = +0.0187 & \pm 0.005 & \pm 0.014 \\
 \delta \omega = +3.12 & \pm 0.65 & \pm 1.95 \text{ (radians)} \\
 \delta T = +0.034 & \pm 0.009 & \pm 0.028 \\
 \delta K = +0.50 & \pm 0.35 & \pm 0.35
 \end{array}$$

The probable errors first given are those resulting from the residuals to the observation equations. An inspection of Fig. 1, however, will show that the component observations of each normal place happen to fall rather symmetrically about the velocity curve. This condition would tend to produce a better agreement of the normal places among themselves than would be expected from the degree of consistency of the individual observations. The probable errors computed from the latter follow the others, and are regarded as more reliable.

In the determination of corrections by means of first differential coefficients, numerical checks should be applied for the purpose of testing the accuracy with which the differential relations hold when the finite corrections are substituted for the differentials. In the present case this is done by computing the velocities in the line of sight $\left(\frac{dz}{dt}\right)$, from both elements I and II, for the times corresponding to the normal places; then the values of

$$\delta' \left(\frac{dz}{dt}\right) = \left(\frac{dz}{dt}\right)_{\text{II}} - \left(\frac{dz}{dt}\right)_{\text{I}}$$

should agree within reasonable limits with those of

$$\delta \left(\frac{dz}{dt}\right)$$

obtained from the observation equations by omitting the first and

last terms. These quantities are tabulated in columns 6 and 7 of Table II. The discrepancy in the case of (8) arises from the rapid variation of the coefficient b with the time: but it is not large enough sensibly to affect the result.

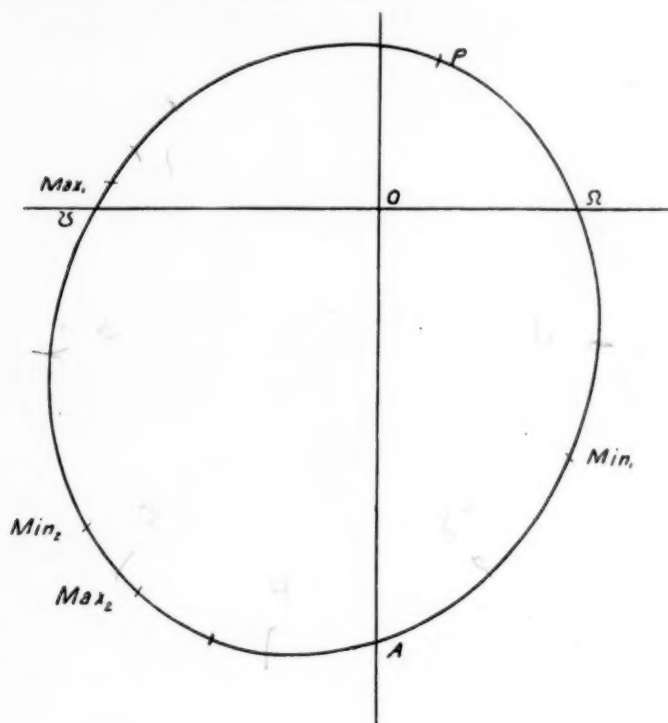


FIG. 3.

The following are the final elements:

ELEMENTS II.

$V = -14.16 \text{ km} \pm 0.17 \text{ km}$ velocity of system in line of sight.

$e = 0.489 \pm 0.014$

$\omega = 68.91^\circ \pm 1.95^\circ$

$T = 6.210^d \pm 0.028^d$

$K = 20.59 \pm 0.35, \frac{f}{V \rho} \sin i.$

As being of possible interest in connection with the matter of the star's light variation, the light curve due to Schur is given in the lower part of Fig. 2. A diagram of the orbit is also appended. The positions of the star at principal maximum and minimum are indicated by Max. and Min., those at the secondary phases by Max.₂ and Min.₂. *P* indicates the position of periastron.

I wish to acknowledge the advice and assistance of Professor Campbell, given during the progress of this research.

TABLE I.

Obs. No.	Plate No.	Date	Days after Light Max.	Vel.	Wt.	O—C Elem. I	O—C Elem. II
1	781 B	June 21.882	1.662	—25.3	$\frac{2}{3}$	—1.2	—1.3
2	789 C	27.901	0.504	—30.6	$\frac{2}{3}$	—0.9	—0.4
3	791 B	28.887	1.490	—25.2	1	—0.1	—0.2
4	795 D	July 4.930	0.357	—30.4	1	—0.3	+0.3
5	799 C	5.858	1.285	—24.5	$\frac{2}{3}$	+1.7	+1.7
6	810 C	12.903	1.153	—27.7	$\frac{2}{3}$	—0.8	—0.7
7	26 B	19.867	0.941	—28.6	$\frac{2}{3}$	—0.6	—0.5
8	46 B	26.821	0.719	—28.8	$\frac{2}{3}$	+0.1	+0.4
9	54 A	Aug. 2.729	0.450	—29.8	$\frac{2}{3}$	+0.1	+0.6
10	57 A	6.798	4.519	—0.3	$\frac{2}{3}$	+2.2	+2.3
11	61 A	7.727	5.448	+9.0	$\frac{2}{3}$	+0.5	+0.5
12	{ 62 B*	7.791	5.512	+11.4 } +11.4 }	$\frac{1}{3}$	+2.4	+2.4
13	64 A	8.724	6.445	—14.1	$\frac{2}{3}$	+1.7	+2.0
14	68 A	15.728	6.273	—9.2	$\frac{2}{3}$	—2.3	—2.2
15	77 B	19.753	3.121	—14.6	$\frac{2}{3}$	+0.2	0.0
16	83 B	21.758	5.126	+2.7	$\frac{2}{3}$	—2.0	—1.9
17	86 A	25.685	1.877	—23.6	1	—0.7	—0.9
18	87 B	25.742	1.934	—20.6	1	+2.0	+1.8
19	92 A	26.687	2.879	—16.6	1	—0.1	—0.4
20	{ 95 A*	28.697	4.889	+1.5 } +1.7 }	$\frac{2}{3}$	—0.1	0.0
21	901 A	29.672	5.864	+8.0	1	—0.6	—0.7
22	99 A	30.676	6.868	—28.8	1	—1.2	+0.3
23	32 D	Sept. 9.726	2.565	—19.5	1	—0.8	—1.1
24	47 A	17.643	3.298	—12.4	1	+1.1	+0.8
25	48 A	18.671	4.334	—6.7	$\frac{2}{3}$	—2.1	—2.2
26	{ 86 A*	Oct. 9.640	3.773	—13.5 } —14.3 }	1	—4.8	
27	94 A	10.637	4.770	+0.8	$\frac{2}{3}$	+0.5	+0.7
28	1008 B	17.733	4.690	+0.1	$\frac{2}{3}$	+0.7	+0.9

* Measured and reduced by Professor Campbell.

TABLE II.

No.	T	Obs. used	m	Wt.	$\delta \frac{dz}{dt}$	$\delta \frac{dz}{dt}$	O - C Elem. II
(1)	0.43	2, 4, 9	-0.36	2	-0.46	-0.43	+ .13
(2)	0.94	6, 7, 8	-0.43	2	-0.08	-0.06	-.31
(3)	1.48	1, 3, 5	+0.13	2	+0.16	+0.18	+ .01
(4)	2.12	17, 18, 23	+0.17	3	+0.31	+0.32	-.09
(5)	3.10	15, 19, 24	+0.42	3	+0.35	+0.35	+ .13
(6)	4.51	10, 25, 28	+0.27	2	+0.06	+0.07	+ .26
(7)	4.93	16, 20, 27	-0.50	2	-0.07	-0.08	-.36
(8)	5.67	11, 12, 21	+0.27	2	+0.24	+0.18	+ .15
(9)	6.36	13, 14	-0.30	1	-0.17	-0.13	-.11
(10)	6.87	22	-1.20	1	-1.22	-1.23	+ .09

LICK OBSERVATORY,
January 1899.

ON THE SCALE OF KIRCHHOFF'S SOLAR SPECTRUM.¹

By J. HARTMANN.

THE remarks on Kirchhoff's spectroscopic apparatus communicated by Professor Vogel to the physical-mathematical section on February 17 of this year have again excited interest in the historic instrument with which Kirchhoff made the observations for his map of the solar spectrum. This, therefore, seems an appropriate time to settle finally a question which has often been raised as to the Kirchhoff spectrum, namely, the meaning of Kirchhoff's scale-divisions and their conversion into wave-lengths.

Kirchhoff placed above his drawing of the spectrum a millimeter scale with an arbitrary zero-point for the sole purpose of convenience in designating the lines that were entered on the map. No simple relation exists between the scale-readings of the separate lines and their wave-lengths, because, as Kirchhoff expressly stated, the prisms were set now more, and again less, accurately for the minimum of deviation of the rays to be measured. As the necessity developed of introducing the natural scale of wave-lengths in place of Kirchhoff's arbitrary divisions, a purely empirical mode of conversion therefore had to be employed. By introducing the wave-lengths of certain lines as measured elsewhere, and assuming that Kirchhoff's scale was at least continuous, the wave-lengths of the remaining lines could be interpolated graphically or by calculation. The large number of attempts to solve the problem of accurately converting Kirchhoff's scale-readings into wave-lengths is explained by the wide employment of the Kirchhoff spectrum—on account of the accuracy of the map, which was not surpassed for several decades after its publication—as well as by the difficulty of the problem itself.

¹ Presented by Professor Vogel at the session of the physical-mathematical section of the Berlin Academy on November 17, 1898. Translated from the *Sitzungsberichte* of the Academy.

The first work in this field was published by W. Gibbs in 1867.¹ He drew an interpolation-curve on a large scale, based on 111 lines measured by Ångström and Ditscheiner and reduced to Ångström's wave-length of the D line, from which he took out the wave-length corresponding to every tenth Kirchhoff scale-division K . Inasmuch as a sufficient number of normal lines was lacking in the portion from A to C, his table included only the region from $K=700$ to $K=2870$, corresponding to the wave-lengths $655\mu\mu$ to $430\mu\mu$. In a second paper² Gibbs employed numerical instead of graphical interpolation, using as an interpolation formula the series

$$\lambda = a + bK + cK^2 + dK^3 + \dots$$

As he had found that the whole Kirchhoff spectrum could not be represented by a single series of that sort at one time, he divided the spectrum into twelve parts, apparently limiting these sections by the purely superficial rule that each should contain ten normal lines. It now appeared that the interpolation formula took very different shapes for the different sections, as is easily seen from the following table:

Section	Kirchhoff		Length	Highest power of K	Value of c
	Beginning	End			
1	694.1	877.0	42.4 $\mu\mu$	3	+1.0
2	877.0	1135.1	42.7	4	-5.6
3	1135.1	1303.5	20.4	3	+1.5
4	1303.5	1421.5	13.6	3	+2.2
5	1421.5	1577.6	14.4	3	-0.7
6	1577.6	1750.4	12.8	3	+0.9
7	1750.4	1920.2	11.5	3	-0.2
8	1920.2	2067.1	11.2	3	-4.2
9	2067.1	2250.0	16.3	1	0.0
10	2250.0	2547.2	21.1	1	0.0
11	2547.2	2721.6	11.4	3	-0.5
12	2721.6	2869.7	8.4	4	+2.1

¹"On the construction of a normal map of the solar spectrum." *Am. Jour. Sci.* (2) 43, 1, 1867.

²"On the measurement of wave-lengths by the method of comparison." *Am. Jour. Sci.*, (2) 45, 298, 1868.

As the next to last column shows, the interpolation curve is a straight line in the ninth and tenth sections, a parabola of the fourth order in the second and twelfth sections, and a parabola of the third order in the remaining sections. The frequent change of the sign of c shows further that the curve is very irregular in its different sections, being sometimes convex and sometimes concave upward. Since Kirchhoff's measures are very well represented by this curve, we must conclude that the dispersion of Kirchhoff's spectrum exhibits irregular and very marked variations.

In a third paper,¹ from which the above table was taken, Gibbs gave certain corrections to his previous values. He further showed that in the ninth and tenth sections the observations are no better represented by parabolas of the second, third, and fourth orders than by a straight line, whence he concluded that the parabolas of higher orders given by the equation

$$\lambda = a + bK + cK^2 + dK^3 + \dots$$

are not suitable for representing these parts of Kirchhoff's spectrum. In this paper Gibbs employs his formulæ finally to calculate the accurate wave-lengths on Ångström's system of all the lines which Kirchhoff observed in terrestrial spectra. We can safely say that Kirchhoff's measures are reduced to wave-lengths as accurately in these very careful researches by Gibbs as was in any way possible with the means available at that time.

A less favorable opinion must be expressed on the contemporaneous papers by Airy.² He employed the same series as an interpolation formula, but he assumed that the whole of Kirchhoff's spectrum could be represented by a single formula of that sort, and therefore based his computations solely on the five normal lines absolutely necessary for the determination of the five constants of the parabola of the fourth order, the lines being the five measured by Fraunhofer, C, D, E, F, and G. When he

¹"On the wave-lengths of the spectral lines of the elements." *Am. Jour. Sci.*, (2) 47, 194, 1869.

²"Computation of the lengths of the waves of light corresponding to the lines in the dispersion spectrum measured by Kirchhoff." *Phil. Trans.*, 158, 29, 1868.

learned, during his calculations, of Ditscheiner's measures, he adopted from them the new determinations of the wave-lengths of the five lines mentioned, but instead of connecting his interpolation with all the 107 Kirchhoff lines measured by Ditscheiner he added the line B as a sixth normal, at the same time introducing the fifth power of K into his interpolation formula. He then computed with this formula the wave-lengths of all the lines of Kirchhoff's spectrum. A comparison of his values with the wave-lengths of numerous lines directly measured by Ångström and by Ditscheiner now convinced Airy that the wave-lengths calculated from his interpolation formula gave the true values for the six normal lines only, but were extremely in error between each two normal lines, the error reaching 145 Kirchhoff units between F and G.

Airy now sought for an explanation of these large errors, and since they did not exhibit any jumps between the normal lines, which he supposed would have been the indication of a change in the setting of Kirchhoff's prisms, he concluded that the cause of the discrepancies between calculation and observation could be found only in one of the three following points: First, the interpolation formula employed might be unsuitable for the purpose; secondly, a change of the method of observation might have been made by Kirchhoff in case of just the six normal lines employed; thirdly, this change might have been made by Ditscheiner and by Ångström. Airy did not consider as probable the two last named explanations, and, therefore, he held the interpolation formula responsible for all errors. In several places in his paper he states that he considers Kirchhoff so good an observer that surely no large error, probably no noticeable error in the map of the spectrum could have arisen from the changes in the setting of the prisms. We see that this view is in direct contradiction with the conclusions drawn from Gibbs' figures, and I shall show below that Airy's assumption was not at all appropriate to the case. Several years later, convinced of the inadequacy of his method of interpolation, Airy corrected his former results by a graphical process, and gave a

new table¹ of the wave-lengths of all of Kirchhoff's lines. This table may be considered as about equal in value to that of Gibbs.

A means of converting Kirchhoff's scale readings into wave-lengths was given, in a somewhat different form, by Stoney.² He proposed that the scale of wave-lengths be drawn directly along with Kirchhoff's spectrum, and, since the intervals of the new graduation are not equal, he gave the positions on Kirchhoff's earlier millimeter scale where the division marks should be entered. His figures are based upon fifty-five of Ångström's lines, and seem to fulfill their purpose entirely.

Two years after Stoney, but it seems entirely independently of him, Thalén³ issued a quite analogous table, but like Gibbs he did not treat the part of the spectrum most difficult to convert, from A to C. On the other hand, Thalén's table extends beyond G, the limit of Kirchhoff's map, to H, referring for this part of the spectrum to a continuation of Kirchhoff's map published by Thalén himself in 1865.⁴ The irregularity of Kirchhoff's scale is clearly brought out by these tables of Stoney and Thalén. Note for instance, the differences in the following portion of Thalén's table :

A	K	Diff.	A	K	Diff.
4300	2867.2		4900	2029.9	117.7
4400	2693.0	174.2	5000	1894.7	135.2
4500	2538.0	155.0	5100	1748.0	146.7
4600	2396.7	141.3	5200	1611.0	137.0
4700	2267.4	129.3	5300	1489.2	121.8
4800	2147.6	119.8	5400	1393.8	95.4

In the two editions of his *Index of Spectra*⁵ Watts gave two different conversions of all the lines observed by Kirchhoff in

¹ "Corrections to the computed lengths of waves of light published in the Philosophical Transactions of the year 1868." *Phil. Trans.*, **162**, 89, 1872.

² "On the physical constitution of the Sun and Stars." *Proc. R. S.*, **17**, 17, 1868-9.

³ "Mémoire sur la détermination des longueurs d'onde des raies métalliques." *Ann. de Chim. et de Phys.* (4) **18**, 211, 1869.

⁴ *K. Vetenskaps-Akademiens Handlingar*, Stockholm, 1865.

⁵ First edition, London, 1872; second edition, Manchester, 1889.

the spectra of terrestrial substances. In the earlier edition the wave-lengths are given to four places, in the later edition to five places, being derived by means of graphical interpolation from Ångström's absolute determinations. When he carried out a similar conversion for Huggins' spectrum it appeared that the curve of interpolation for the latter was in fact smoother, but fitted the individual lines less well than the curve drawn for Kirchhoff's spectrum. Watts concluded from this that Kirchhoff's measures were more accurate individually than those of Huggins, but that the latter formed a uniform system, which is not the case with those of Kirchhoff.

More accurate fundamental determinations of wave-lengths having been meanwhile carried out, Hasselberg¹ published in 1878 a new conversion table quite analogous to that of Gibbs. It is obtained by graphical interpolation from a large number of lines taken from Ångström's *Recherches sur le spectre solaire*. This table probably reaches the limit of accuracy possible by a graphical process in the transformation of Kirchhoff's scale-divisions into wave-lengths of Ångström's system. Hasselberg's table also begins at B, and furnishes no points of reference for the part of the spectrum most difficult to transform, from A to B.

Beside these more extensive researches, which aimed at a reduction of the whole or nearly the whole of Kirchhoff's spectrum, we shall now mention briefly several less important papers in which the wave-lengths were determined for only a limited number of Kirchhoff's lines.

Ditscheiner's² direct measurement of the wave-lengths of 107 of Kirchhoff's lines first deserves mention. In this investigation on which the above researches of Gibbs and Airy are based, Ditscheiner most carefully identified on Kirchhoff's spectrum the lines whose wave-lengths he had measured. For this purpose he measured a large number of lines not only in the diffraction spectrum, but also in the spectrum of a flint glass prism

¹ "Zur Reduction der Kirchhoff'schen Spectralbeobachtungen auf Wellenlängen." *Bull. de l'Acad. de St. Petersbourg*, 25, 131, 1879.

² "Bestimmung der Wellenlängen der Fraunhofer'schen Linien des Sonnenspectrums." *Sitzungsber. der Wiener Akad.* 50, II, 296, 1864.

of 60° angle. The prismatic spectrum thus obtained could be directly compared with Kirchhoff's map, thereby rendering possible a sure identification of the lines.

A similar series of observations, which have received little attention, was published by Weinhold¹ in 1869. In measuring the wave-lengths of 128 of Kirchhoff's lines he employed the interference bands which arise in the prismatic spectrum, parallel to the Fraunhofer lines, when the light is made to interfere before entering the slit by reflection from a sheet of mica.²

Ångström himself only identified a few of his lines on Kirchhoff's map, but tables giving both Kirchhoff's designation and Ångström's wave-lengths may be found in works of various authors, as for instance d'Arrest,³ Secchi,³ and Young.⁴ The last of these tables was copied in the text-books on spectrum analysis by Schellen and by Roscoe.

Since the appearance of Rowland's great atlas of the solar spectrum, produced by direct photography and provided with an accurate scale of wave-lengths, it is easy to obtain the wave-length corresponding to every line drawn by Kirchhoff. The identification of Kirchhoff's lines among the much more numerous lines of Rowland's spectrum, offers in general no difficulties; it is indeed really a pleasure to observe the accuracy with which the impression of complicated close groups, for the resolution of which Kirchhoff's apparatus was insufficient, is reproduced in the drawing by the different degrees of blackness and width of the lines. I have carried out this identification for large portions of the spectrum, but I do not publish here a complete catalogue of all of Kirchhoff's lines, for a comprehensive list of that sort would have at present only slight value. I wish here only to investigate how far Kirchhoff's map differs from a correctly drawn prismatic spectrum, how the discrepancy arose, and how

¹ "Ueber eine vergleichbare Spectralscale." *Pogg. Ann.*, **138**, 417, 1869.

² "Undersogelser over de nebulose Stjerner." Kopenhagen, 1872, p. 28.

³ "Die Sonne," deutsch von Schellen. Braunschweig, 1872. Vol. I, p. 246.

⁴ "Catalogue of bright lines in the spectrum of the solar atmosphere." *Am. Jour. Sci.* (3) **4**, 356, 1872.

accurate wave-lengths may nevertheless be calculated in a simple way from Kirchhoff's scale-readings.

By a correctly drawn dispersion spectrum is to be understood that which is obtained when every single line is observed at minimum deviation and the angles of deviation are then drawn on a linear scale. I shall designate such a spectrum briefly as an ideal dispersion spectrum, and I mention as an illustration the "spectrum of the Sun with weak dispersion" drawn by Professor Müller, in Plate 34, of the second volume of the Potsdam Publications. The ideal spectrum cannot be directly observed; our immediate perception takes in only the somewhat differently constituted spectra which are obtained when the position of the prism is fixed with reference to the incident ray and the setting is for the minimum of deviation for any line n . I shall call such a spectrum the n -spectrum. With a spectroscope adjusted to the minimum deviation of the D lines we shall get with this notation a D-spectrum; with adjustment for the minimum of deviation of F or $H\gamma$ an F or $H\gamma$ -spectrum, respectively. A clear distinction between these two kinds of dispersion spectra is necessary not only here, but it will also contribute to clearness in many other cases. In both spectra the position of a line can be rigorously calculated from its index of refraction, but the formulæ are totally different in the two cases. Moreover the wave-length of a line can be directly calculated from its position in both spectra by a simple dispersion formula which I have brought forward in a special paper (Potsdam Publications, Vol. XII, No. 42).

According to the clear explanation of the mode of conducting his observations, given by Kirchhoff himself, his spectrum was not measured in one piece, but the prisms were differently adjusted for the different parts. In Kirchhoff's spectrum we therefore have before us a succession of adjacent n -spectra.

First a comparison of Kirchhoff's map with the ideal spectrum of his spectrometer is interesting. From the angles of the prisms and indices of refraction given in the paper by Professor Vogel, mentioned at the beginning, we derive the

minimum deviations of the following table, which can then be converted into Kirchhoff's units by means of the relation $1^\circ = 295.83 K$.¹

Line	Index <i>n</i>	Minimum of deviation
B	1.6093	$140^\circ 29' 20'' = 41500 (H\gamma) K$
C	1.6110	$140 \ 56 \ 16 = 41693.6$
D	1.6158	$142 \ 12 \ 28 = 42069.4$
δ_1	1.6230	$144 \ 7 \ 10 = 42635.0$
F	1.6275	$145 \ 19 \ 10 = 42990.0$
$H\gamma$	1.6375	$147 \ 59 \ 58 = 43782.8$

The difference between the successive numbers of the last column gives the extent which the respective portions of the ideal spectrum would have for Kirchhoff's apparatus. A comparison of these sections with the corresponding parts of Kirchhoff's map furnishes the following table:

Section	Ideal spectrum	Kirchhoff's spectrum	Difference <i>K-I</i>
B-C	132.8 <i>K</i>	100.6 <i>K</i>	-32.2 <i>K</i>
C-D	375.8	310.7	-65.1
D- δ_1	565.6	629.3	+63.7
δ_1 -F	355.0	445.9	+90.9
F- $H\gamma$	792.8	716.2	-76.6

Kirchhoff's spectrum therefore does not correspond at any point to the dispersion obtained with the prisms accurately set for the minimum deviation of just the rays being observed. The scale of the map is too large in the middle portion of the spectrum from D to F, which was observed by Kirchhoff himself, and is too small in the exterior portions measured by Hofmann. The irregularity of Kirchhoff's scale is so considerable that, for instance, the portion of spectrum from B to D

¹ Kirchhoff's scale division is equal to one eighteenth of a revolution of the measuring screw, which is almost tangentially attached and which has a head divided into 180 parts. The angular motion of the observing telescope corresponding to this revolution is in consequence of the mode of attachment of the screw not constant, but variable by about 1 per cent. of its whole value. The number given above is the mean, obtained when the whole length of the screw is used.

should be drawn 45 per cent. larger, or almost half of the whole length greater, if it is to be reduced to the same scale as the stretch from D to F.

The direct measurement of the ideal spectrum—that is, the observation of every line at its minimum deviation—would have been so extremely laborious with his spectroscope that Kirchhoff was fully justified in contenting himself with only approximating this spectrum. In order to prevent misunderstanding he therefore himself called attention to the irregularity of his scale. As a matter of fact, it would have been much more convenient, both in making the measures and in reducing them to wave-lengths, if the prisms had been left unchanged in position for the whole spectrum. It appears that the length of the measuring screw as well as the aperture of the prisms and observing telescope was sufficient for measuring the whole Kirchhoff spectrum from A to G in one piece. In this way, with the prism-train accurately adjusted for the minimum deviation of F, I have repeated the measurements of the principal lines of the whole spectrum, and give a comparison of this series of observations with Kirchhoff's scale-readings in the following table:

Line	F-spectrum	Kirchhoff's spectrum	Difference ($K-F$)
A	401.2 K'	401.2 K'	0.0 K'
B	627.3	593.5	— 33.8
C	744.0	694.1	— 49.9
D ₁	1073.8	1002.8	— 71.0
D ₂	1077.2	1006.8	— 70.4
b_1	1615.4	1634.1	+ 18.7
b_4	1631.1	1655.7	+ 24.6
F	1968.8	2080.0	+ 111.2
H γ	2797.3	2796.2	— 1.1
G	2864.1	2854.4	— 9.7

We see that the whole length of Kirchhoff's spectrum from A to H γ agrees exactly with that of the F-spectrum; there is also a line between D₂ and b_1 which falls at its right place; all the preceding lines are too far toward the red, and all the succeeding lines too far toward the violet, whence again it follows

that the middle part of the spectrum was drawn too large, and the beginning and the end too small. If we again calculate the extent of the same sections as in the previous table we get the following results :

Section	F-spectrum	Kirchhoff's spectrum	Difference ($K-F$)
B-C	116.7 K	100.6 K	- 1.16 K
C-D	331.5	310.7	- 20.8
D- b_2	539.9	629.3	+ 89.4
b_1 -F	353.4	445.9	+ 92.5
F-H γ	828.5	716.2	- 112.3

The deviations from the F-spectrum lie in the same direction and are of the same order of magnitude as in case of the ideal spectrum, furnishing a complete confirmation of what was said above, on comparing Kirchhoff's map with the ideal spectrum.

As has been shown in what precedes, Kirchhoff's spectrum is composed of a number of parts measured with different dispersion, and it is important to determine accurately the extent of these separate parts. A little while ago such an investigation could hardly have been carried out, but since the discovery of the new dispersion formula mentioned above, it no longer presents difficulties. If we denote, as heretofore, Kirchhoff's scale-reading by K and the wave-length by λ , the formula reads :

$$(K - K_0)(\lambda - \lambda_0)^a = c.$$

λ_0 , K_0 , and c are constants, to be determined from the observations; if it is desired to represent the whole spectrum by the formula, a is to be given the value 1.2; if we limit ourselves to the representation of shorter stretches of the spectrum, a may be simply placed equal to 1. As an illustration of the first mentioned application of the formula, we shall convert into wave-lengths the values of K , given above, from my measurement of the F-spectrum. Employing the three lines A, b_1 , and G, we get the formula :

$$(\lambda - 225.10)^{1.2} = \frac{[6.332465]}{K + 738.3},$$

from which the following wave-lengths are calculated. For purposes of comparison I have appended Rowland's wave-lengths.

Line	K	λ calculated	λ Rowland	Difference
		$\mu\mu$	$\mu\mu$	$\mu\mu$
A (space).....	401.2	761.88	761.90	— 0.02
B (edge).....	627.3	686.72	686.75	— 0.03
C.....	744.0	656.23	656.30	— 0.07
D_1	1073.8	589.77	589.62	+ 0.15
b_1	1615.4	518.37	518.38	— 0.01
F.....	1968.8	486.10	486.15	— 0.05
$H\gamma$	2797.3	434.03	434.06	— 0.03
G.....	2864.1	430.80	430.80	0.0

We see that the formula gives the correct wave-lengths for the whole spectrum.

A precisely similar conversion must now be possible for Kirchhoff's spectrum, except that a special formula must apply to each of the separate sections measured with unchanged setting of the prisms. On account of the limited extent of the separate parts it is here permissible to simply place $a = 1$. The points at which changes were made in the adjustment of the apparatus betray themselves by the fact that the formula which had previously well represented the observations there suddenly begins to be inapplicable. I carried out the investigation by first obtaining the exact values of the wave-lengths of a large number of Kirchhoff's lines by direct identification with the photographic atlases of Rowland and Higgs. Then beginning with the first section of Kirchhoff's maps (extreme red), the interpolation formula

$$\lambda = \lambda_0 + \frac{c}{K - K_0}$$

was first closely applied to a short portion of the spectrum. Proceeding to shorter wave-lengths with the formula thus obtained the wave-lengths were calculated from Kirchhoff's scale-readings for all the lines identified. From the agreement of these values with Rowland's accurate wave-lengths it was then easy to see whether it was possible, by a slight alteration of the constants, to extend the application of the formula previously

employed to a longer stretch of spectrum, or whether the representation of two adjacent regions by one and the same formula was impossible. In the latter case it was thus disclosed that a significant change in the dispersion of the apparatus had occurred from a displacement of the prisms.

The fact has been brought out in this way that the whole spectrum drawn in eight strips by Kirchhoff and Hofmann is composed of five parts which differ not inconsiderably in their dispersion as well as in their accuracy. The separate parts have the following extent:

The first section embraces the stretch from A to D, which was drawn by Hofmann on strips 1 and 2. The wave-lengths of the lines, on Rowland's system, are obtained from the formula

$$\lambda = 332.22\mu\mu + \frac{[5.587969]}{K + 500.0}. \quad (1)$$

The second section, extending from D nearly to E ($K = 1500$), was drawn by Kirchhoff himself, and occupies the third and most of the fourth strip. The formula which applies is

$$\lambda = 270.46\mu\mu + \frac{[5.831503]}{K + 1122.4}. \quad (2)$$

The remainder of the fourth strip comprises, with the fifth, the third section, extending to $K = 1940$. The transformation formula reads

$$\lambda = 246.73\mu\mu + \frac{[5.991027]}{K + 1971.8}. \quad (3)$$

The sixth strip, which was also measured by Kirchhoff, extends to $K = 2250$, including the region about the F line, and forms a part by itself. The prisms here stood in quite an erroneous position so that this part is drawn much out of proportion, as is expressed in the following formula:¹

¹For comparison we remark that the corresponding formula, calculated with $\alpha = 1$, which fits the entire F-spectrum given above from A to G within $0.3\mu\mu$, reads

$$\lambda = 252.76\mu\mu + \frac{[5.828844]}{K + 924.4}.$$

The more the constants of a formula differ from this, the more inaccurate was the position of the prisms.

$$\lambda = 891.04\mu\mu - \frac{[6.290971]}{6904.6 - K} \quad (4)$$

The remaining part of the spectrum, given by Hofmann in strips 7 and 8, was observed with a good adjustment of the prisms. The formula reads :

$$\lambda = 248.85\mu\mu + \frac{[5.778426]}{K + 446.0} \quad (5)$$

The following table shows the accuracy with which the wave-lengths can be computed from Kirchhoff's scale-divisions by the five formulæ we have given. The lines given were selected from Kirchhoff's map in nearly equal intervals of 20 to 30 scale-divisions. The first column gives Kirchhoff's scale-reading, the second the exact wave-lengths according to Rowland; the third and fifth columns contain the wave-lengths calculated from the above formulæ, and the fourth and sixth the corresponding differences between observation and computation. The regions for which the different formulæ hold good are separated by horizontal lines, and the formula employed is indicated by the Roman numerals above the computed values of λ .

This table shows very prettily how each formula fits the observations closely within the region of its validity, but beyond its range leads at once to values of the wave-lengths departing systematically from the true values.

The values O—C lead to the following probable errors of the position of a line in Kirchhoff's spectrum :

Section	I	Probable error = $\pm 0.251\mu\mu = \pm 0.93K$	
	II	0.037	0.31
	III	0.017	0.24
	IV	0.090	1.07
	V	0.028	0.42

The large value of the probable error expressed in $\mu\mu$ in case of the first section is chiefly due to the great compression of the region of long wave-lengths in the prismatic spectrum. The errors expressed in Kirchhoff units are nevertheless directly comparable with each other. As would be expected, the uncer-

K	λ (Rowland)	λ (Computed)	O—C	λ (Computed)	O—C
		I			
404.1	760.60 μ	760.53 μ	+0.07 μ		
423.7	751.11	751.44	— .33		
448.4	740.03	740.52	— .49		
470.0	731.87	731.43	+ .44		
489.6	724.08	723.52	+ .56		
513.6	714.84	714.25	+ .59		
540.6	704.01	704.34	— .33		
564.1	695.67	696.12	— .45		
597.4	685.54	685.08	+ .46		
626.1	676.80	676.09	+ .71		
654.3	667.82	667.69	+ .13		
678.6	660.94	660.77	+ .17		
694.1	656.30	656.51	— .21		
714.4	650.88	651.08	— .20		
740.9	643.93	644.27	— .34		
759.3	639.38	639.72	— .34		
786.8	632.78	633.14	— .36		
815.0	626.53	626.69	— .16		
845.7	620.05	619.97	+ .08		
866.2	615.79	615.66	+ .13		
891.7	610.83	610.46	+ .37		
916.3	605.62	605.63	— .01	II	
943.4	600.32	600.50	— .18	607.30 μ	+3.53 μ
969.6	595.69	595.71	— .02	603.23	+2.39
991.9	591.44	591.77	— .33	598.87	+1.45
1002.8	589.62	589.89	— .27	594.76	+0.93
1025.5	586.26	586.06	+ .20	591.33	+ .11
1035.3	584.83	584.44	+ .39		
1058.0	581.66	580.76	+ .90	589.69	— .07
1089.6	577.24	575.82	+1.42	586.32	— .06
1111.4	574.21			584.88	— .05
1130.9	571.53			581.61	+ .05
1151.1	568.84			577.16	+ .08
1174.2	565.90			574.17	+ .04
1193.1	563.42			571.54	— .01
1217.8	560.31			568.87	— .03
1245.6	556.98			565.87	+ .03
1267.3	554.42			563.45	— .03
1287.5	551.98			560.36	— .05
1315.0	548.80			556.96	+ .02
1343.5	545.58			554.36	+ .06
1367.0	542.99			551.98	.00
1394.2	540.07	III		548.80	.00
1425.4	536.77	537.74	+2.33	545.58	.00
1444.4	534.85	535.07	+1.70	542.99	.00
1466.8	532.44	533.47	+1.38	540.04	+ .03
1487.7	530.25	531.60	+0.84	536.74	+ .03
1506.3	528.38	529.88	+ .37	534.77	+ .08
1522.7	527.05	528.36	+ .02	532.48	— .04
1547.2	525.08	527.04	+ .01	530.38	— .13
1573.5	523.00	525.09	— .01	528.54	— .16
1598.9	521.06	523.03	— .03	526.94	+ .11
1623.4	519.16	521.06	.00	524.59	+ .49
		519.19	— .03	522.11	+ .89
				519.76	+1.30
				517.54	+1.62

K	λ (Rowland)	λ (computed)	O—C	λ (computed)	O—C
1647.3	517.39 $\mu\mu$	517.39 $\mu\mu$	0.00 $\mu\mu$		
1662.8	516.24	516.24	.00		
1681.6	514.83	514.85	— .02		
1701.8	513.39	513.38	+ .01		
1733.6	511.06	511.09	— .03		
1762.0	509.10	509.08	+ .02		
1785.0	507.49	507.47	+ .02		
1806.4	506.03	505.99	+ .04		
1830.1	504.44	504.38	+ .06		
1854.9	502.73	502.71	+ .02	IV	
1884.3	500.74	500.76	— .02	501.78 $\mu\mu$	—1.04 $\mu\mu$
1904.5	499.43	499.43	.00	500.21	—0.78
1925.8	498.04	498.05	— .01	498.53	— .49
1939.5	497.15	497.17	— .02	497.45	— .30
1960.8	495.78	495.81	— .03	495.76	+ .02
1994.1	493.05	493.72	— .67	493.07	— .02
2026.8	490.35	491.70	—1.35	490.41	— .06
2058.0	487.84	489.81	—1.97	487.83	+ .01
2080.0	486.15			485.99	+ .16
2103.3	484.05			484.02	+ .03
2136.0	481.07	V		481.23	— .16
2167.5	478.36			478.51	— .15
2184.9	476.85	477.05	—0.20	476.99	— .14
2201.9	475.42	475.59	— .17	475.49	— .07
2222.3	473.70	473.85	— .15	473.68	+ .02
2249.7	471.46	471.57	— .11	471.22	+ .24
2264.2	470.32	470.38	— .06	469.91	+ .41
2278.4	469.16	469.22	— .06	468.62	+ .54
2302.9	467.25	467.26	— .01	466.37	+ .88
2325.3	465.47	465.49	— .02	464.29	+1.18
2354.1	463.31	463.26	+ .05	461.59	+1.72
2379.0	461.35	461.37	— .02		
2406.6	459.28	459.32	— .04		
2422.3	458.16	458.16	.00		
2446.6	456.39	456.41	— .02		
2461.2	455.42	455.36	+ .06		
2497.2	452.88	452.84	+ .04		
2517.0	451.55	451.48	+ .07		
2547.2	449.47	449.43	+ .04		
2565.0	448.24	448.25	— .01		
2585.4	446.87	446.90	— .03		
2602.1	445.77	445.82	— .05		
2627.0	444.25	444.22	+ .03		
2653.2	442.56	442.57	— .01		
2670.0	441.53	441.53	.00		
2693.5	440.06	440.08	— .02		
2721.2	438.37	438.41	— .04		
2744.1	436.99	437.05	— .06		
2774.0	435.29	435.30	— .01		
2800.7	433.77	433.77	.00		
2822.0	432.59	432.56	+ .03		
2841.7	431.51	431.46	+ .05		
2875.2	429.68	429.62	+ .06		

tainty of the measures is somewhat greater in the outer, less easily visible parts of the spectrum than in the middle. The large value of the probable error in the fourth section is alone striking. It looks as if slight changes had frequently occurred in the adjustment of the apparatus, due to jars or changing of the focus of the observing telescope, during the measurement of this, the shortest section of the spectrum. In the best section, the third, the probable error of a line given above reaches almost exactly the value $\pm 0.015\mu\mu$ which Professor Vogel derived from repeated settings on the same line at the middle part of the visible spectrum. The extreme accuracy of Kirchhoff's measurements is thus clearly shown.

THE VARIABLE VELOCITY OF ζ GEMINORUM IN THE LINE OF SIGHT.

By W. W. CAMPBELL.

MR. WRIGHT and I have found that the well-known variable star ζ Geminorum ($\alpha = 6^h 58^m$, $\delta = +20^\circ 43'$) has a variable velocity in the line of sight. Three spectrograms have been obtained, yielding the following velocities with reference to the solar system.

1898 November 11	- - - - -	$V = +20.0$ km
1899 January 17	- - - - -	$- 6 \pm$
January 18	- - - - -	$+ 7 \pm$

The last two plates are underexposed on account of dew on the object-glass of the telescope, and the results obtained from them are not to be used in subsequent discussions of the motion.

LICK OBSERVATORY,
January 19, 1899.

ON THE APPLICATION OF INTERFERENCE PHENOMENA TO THE SOLUTION OF VARIOUS PROBLEMS OF SPECTROSCOPY AND METROLOGY.¹

By A. PEROT and CHARLES FABRY.

INTERFERENCE phenomena permit us to refer determinations of length to a very small unit (of the order of $\frac{1}{2}$ micron), the wave-length of a luminous radiation; for this reason the use of these phenomena is at once suggested when it is a question of measuring very small thicknesses or very small changes or differences in thickness. To make evident the services which interference methods may render in this direction, it suffices to mention the investigations of Fizeau on expansion, those of M. Cornu on the elastic change of figure of solid bodies, and the methods devised by M. Laurent for testing optical surfaces.

The extreme minuteness of the wave-length introduces certain difficulties when it is desired to apply interference methods to greater lengths. The measurement will involve at the outset the determination of the number of times the length to be measured contains the wave-length, *i. e.*, the integral part of the number which represents the quantity to be measured in terms of the chosen unit; in practice, this measurement will consist in the determination of the *order* of a fringe. Although a *whole number* is in question, its determination may give rise to some difficulty if it amounts to some tens or hundreds of thousands. This operation being supposed completed (and it must involve no error), the measure will be susceptible of extreme precision, on account of the very minuteness of the chosen unit; it will only remain to determine a fraction of a wave-length, and even if this fraction is found with a rather rough degree of approximation, the quantity to be measured will be known with very great precision. It will often happen that it is double the length

¹ *Bulletin Astronomique*, 16, 5, January 1899.

to be measured which is determined directly in wave-lengths, which will double the precision of the measures.

Knowing how to measure a given distance in wave-lengths, we can compare any two lengths by measuring them successively in terms of this unit. We can also compare with a high degree of precision the wave-lengths of two given radiations, by comparing the same length with each of these wave-lengths. Finally, we can distinguish very small differences of wave-length, and consequently separate close lines in the spectrum, thus permitting the spectroscopic study of a group of lines.

These various applications of interference phenomena presuppose the employment of a light-source such that interference can be obtained with great differences of path; this requires that the light be strictly monochromatic, corresponding to a single and well-defined vibrational motion. It is clear, moreover, that if this condition were not satisfied no precise measurement in wave-lengths would be possible, since the light employed would not have a *single* well-determined wave-length. At the present time it is easy to produce almost absolutely monochromatic¹ radiations, thanks to the light-sources brought into use by Professors Michelson and Morley, which consist, as is well known, of metallic vapors rendered luminous by the discharge of an induction coil.

Further, a suitable interference apparatus is required.

We propose to give a brief account of a part of the investigations which we have made for the purpose of solving the various problems just enumerated, employing an interference apparatus having special properties which will be described at the outset.

I. FRINGES FROM SILVERED PLATES.

The ordinary forms of interference apparatus divide each incident wave into two waves capable of interfering. Each point

¹ The radiations employed up to the present time, excepting the red line of cadmium, are not single, but are composed of several closely grouped lines, one of which is much brighter than the rest; the numbers given for the wave-lengths refer to these predominant lines.

in the focal plane of the observing telescope—the imaginary observation screen—thus receives, from each point of the light-source, two vibratory motions having a difference of path Δ . In order that the phenomenon may be distinct, it is necessary

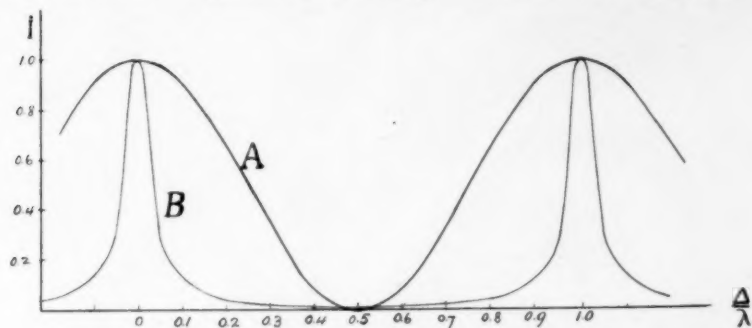


FIG. 1.

that Δ have a single value at every point of the screen, *i. e.*, that every pair of waves reaching a given point have the same difference of path. Supposing this *condition of perfect distinctness* to be satisfied, the luminous intensity will vary from one point to another in the focal plane; it is a function of Δ alone: the curves of equal luminosity are represented by the general equation $\Delta = a$ constant. The maxima are defined by $\Delta = K\lambda$, and the minima by $\Delta = K\lambda + \frac{\lambda}{2}$, λ being the wave-length of the light employed and K a whole number. If we suppose that the two interfering waves have the same intensity (as is ordinarily the case), the minima are zero; the curve which gives the luminous intensity I as a function of Δ is a sinusoid (Fig. 1, curve A). The fringes consequently have the appearance of bright bands, separated by dark bands with ill-defined edges; the passage from maximum brightness to the neighboring minimum is gradual and without abrupt change.

The phenomenon assumes a wholly different aspect if the apparatus, instead of dividing each wave into two, separates it into a very great number having differences of path which are in arithmetical progression, such that the differences of path with

respect to one of them are $\Delta, 2\Delta, 3\Delta, \dots$. A grating is a familiar example of such an apparatus, in which the effect of the superposition of all these waves is known. When Δ is a whole number of wave-lengths, there is accordance of *all* the interfering waves, and consequently a light maximum; but if $\frac{\Delta}{\lambda}$ differs ever so little from a whole number, among all these waves there are some whose difference of path as compared with the first is far from a whole number, and which consequently considerably diminish the resulting intensity. The intensity thus falls off very rapidly away from the maximum, and the phenomenon consists of bright lines which are very narrow as compared with the dark interval which separates two successive maxima. The fineness of these bright lines will be the greater as the number of interfering waves increases.

A phenomenon of this character may arise in certain kinds of interference apparatus, on account of the multiple reflections

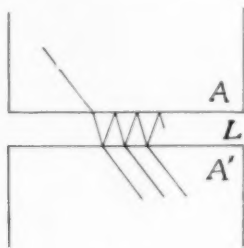


FIG. 2.

which the light may experience. For example, let L be a thin film of air bounded by two transparent surfaces A and A' (Fig. 2). An incident wave will give rise to an infinite number of emergent waves which have respectively undergone 0, 2, 4, \dots , $2n$, \dots reflections, and which have, with respect to the first, differences of path 0, 2Δ , \dots , $n\Delta$, \dots ($\Delta = 2e \cos i$, i being the angle of incidence in the layer of air and e the thickness of this layer). In the case where the surfaces A and A' are simple surfaces of glass, the intensities of these waves decrease very rapidly, on account of the small reflecting power

of glass (about $\frac{1}{20}$); beyond the third they are wholly negligible. The case is different if the glass surfaces A and A' are lightly silvered; by this means it is possible to give them a very high reflecting power, at the same time leaving them a sufficient degree of transparency to permit an appreciable quantity of light to traverse the system. The intensities of the successive waves then decrease in a geometrical progression, the ratio of which does not differ much from unity, and the superposition of an infinite number of these waves gives a result analogous to that obtained with a grating.

Further calculation of the luminous intensity resulting from the superposition of all these waves leads to the expression

$$I = I_0 \frac{1}{1 + \frac{4f}{(1-f)^2} \sin^2 \pi \frac{\Delta}{\lambda}},$$

I_0 being a constant (intensity of the maxima), and f the reflective power of each of the surfaces A and A' . If f differs but little from 1, $\frac{4f}{(1-f)^2}$ is very great; for example, if $f = 0.8$ this fraction is equal to 80, and the expression for I becomes

$$I = \frac{I_0}{1 + 80 \sin^2 \pi \frac{\Delta}{\lambda}}.$$

When $\frac{\Delta}{\lambda}$ is a whole number we have $I = I_0$; but if $\frac{\Delta}{\lambda}$ differs ever so little from a whole number, I becomes almost equal to zero, on account of the term $80 \sin^2 \pi \frac{\Delta}{\lambda}$ in the denominator. The curve B (Fig. 1) represents the law of variation of I as a function of Δ .

Thus a layer of air bounded by two lightly silvered surfaces gives, when examined *by transmission* in monochromatic light, a system of fringes in which the bright part is very narrow as compared with the dark interval which separates two consecutive fringes; the small quantity of light which the system allows to pass is distributed in very narrow bright lines. This effect is

the more pronounced as the reflecting power f becomes more nearly equal to unity; now the reflecting power of silvered glass increases with the thickness of the silver film, and approaches that of the compact metal, which is about 0.90; but at the same time the quantity of light absorbed by the silver films increases. If this absorption did not exist, the intensity I_0 of the maxima would be always equal to that of the incident light; the existence of the absorption limits the thickness of the silver films that can be employed, and this thickness must depend upon the intensity of the light at command. In fact, when the light is fairly intense, it is possible to obtain fringes, the bright part of which does not occupy at the most more than $\frac{1}{20}$ of the interval which separates two consecutive fringes.

In addition to the characteristics described above, fringes from silver films possess the properties of fringes from ordinary isotropic films, and can be examined under the same conditions. It is always necessary to respect the condition of perfect distinctness, *i. e.*, that the value of Δ must be invariable for every point in the observation plane. The observation can be made in two simple ways, the choice of which will be governed by circumstances.

1. *In parallel light normal to the film*; where $i=0$ and $\Delta=2e$. A system of fringes is obtained which describe the curves of equal thickness of the film and whose form essentially depends upon the form of the limiting surfaces. This mode of observation is very convenient when the thickness e is small; it then suffices to have the beam utilized approximately parallel and normal to the film in order to obtain a system of fringes localized in the film (Newton's rings; fringes of thin plates). When great differences of path are reached it is necessary to employ a rigorously parallel beam, without which the variously inclined waves would give scattered fringes, and the phenomenon would be rendered indistinct. The second mode of observation is free from all difficulties of this kind.

2. *In convergent light*, the layer being limited by two plane parallel surfaces. The thickness e is then perfectly constant,

and the difference of path $\Delta = 2e \cos i$ depends only on the angle of incidence i . The fringes are observed by means of a telescope focused for parallel rays. To every point in the focal plane of the telescope there thus corresponds a single value of i , and consequently a single value of Δ ; the conditions of perfect distinctness are thus realized, and a system of rings centered on the normal to the layer is obtained. The expression for Δ may be written, when it is remembered that the field of the telescope is of small extent and that consequently i is small,

$$\Delta = 2e - e i^2.$$

Δ decreases proportionally to i^2 ; the diameters of these rings obey the same law as those of Newton's rings, but with the difference that it is at the center ($i=0$) that Δ attains a maximum. Moreover, these rings at infinity have the appearance of very fine lines, commonly seen in fringes from silvered films.

This second mode of observation is especially advantageous in the case of great differences of path. In order to observe these fringes, two plates of glass, each having a silvered plane face, should be employed. These surfaces, which face one another, must be made exactly parallel; it is convenient to be able to change their distance apart without destroying this parallelism, so that, without readjustment, it may be possible to observe the rings corresponding to various values of the difference of path.

The apparatus employed, which we call an *interference spectroscope*, essentially consists of two plane plates of silvered glass placed vertically. One of these plates, L , carried by an old theodolite, can be given a wide range of displacement in azimuth, and small parallel displacements by means of a strong iron stirrup which can be bent by distending a rubber bag filled with water placed inside the stirrup; this bag is connected, like the two others referred to below, by a rubber tube to a vessel filled with water, the level of which can be varied; it is thus possible to produce a displacement of a few microns by a motion as slow as may be desired, and without lost motion.

The other plate, L' , can be given small adjustments in azimuth

and large parallel displacements. It is carried normally at the end of a horizontal strip of steel, 5mm in diameter and 10cm long, rigidly fixed at the other end, against which two bags filled with water press in two directions at right angles to one another; it is thus possible to obtain through flexure of the steel strip, very small angular displacements which are produced without in any wise disturbing the apparatus. The steel strip is supported by a carriage having the form of a triangular prism with horizontal edges; two strips of St. Gobain glass are cemented to the two lower faces, which meet at a right angle; these rest on ways also made of glass strips cemented to a wood base. The carriage may thus be given a parallel displacement, in which it will be perfectly guided if not subjected to any lateral pressure. To effect this the carriage can be pushed, in either direction, by two points attached to two other auxiliary carriages; the principal carriage has a little play between the two points, and consequently can be pushed by only one of them. The two auxiliary carriages can be moved together by means of a screw connected with them, which passes through a nut that can be turned by hand when a rapid displacement is desired, and by means of a tangent screw when a slower motion is needed. A single turn of this screw corresponds to a displacement of 3μ or 4μ . The whole apparatus, carried on a thick plank, is suspended in the air by rubber rings to protect it from vibration, which would render the fringes invisible.

The adjustment of the parallelism of the plates is effected by moving the plate L for approximate adjustment, and L' , through flexure of its support, for the final adjustment. Large parallel displacements are produced by moving the nut on the thread, and small ones by bending the stirrup which carries the plate L . Accidental displacements, caused by imperfections in the guides along a distance of several centimeters, are so small that the rings do not disappear, and they can be brought back to their normal appearance by bending the support of L' .

A divided scale, attached to the plate-carriage, and read by a fixed microscope, gives a means of measuring approximately the

displacement of the carriage, and consequently the distance between the silvered surfaces.

II. PHENOMENA PRODUCED WHEN THE INCIDENT LIGHT IS COMPOSED OF TWO MONOCHROMATIC RADIATIONS.

Thanks to the fineness of the bright fringes, if several radiations simultaneously enter an interference apparatus with silvered plates, the systems of fringes corresponding to these several radiations are not confused, but may be seen in juxtaposition. Let us consider what occurs when only two radiations are employed; later certain applications of these phenomena will be given.

Let there be a first radiation, red, for example, of wave-length λ . Consider a point in the observation plane for which the difference of path of the first two interfering waves is Δ .

The quotient $p = \frac{\Delta}{\lambda}$ is the difference of path expressed in wave-lengths. This quotient plays a fundamental part in the phenomenon: each integral value of p corresponds to a bright fringe, of which this number expresses the *order*. To avoid circumlocution, we will call p the *order of interference* corresponding to the difference of path Δ ; for the radiation λ a definite value of p corresponds to every point in the observation plane.

If we have in addition a second radiation of wave-length λ' , yellow, let us say, ($\lambda' < \lambda$), it will also give a system of fringes, in which the bright fringes will be defined by the integral values of the quotient $p' = \frac{\Delta}{\lambda'}$. These two systems of bright lines belong, moreover, to the same family of curves, the general equation of which is $\Delta = \text{const.}$, but the yellow correspond to values of Δ which are multiples of λ' , and the red to multiples of λ . In passing from one red fringe to the next, Δ increases by λ ; it increases only by λ' in passing from a yellow fringe to the following one, which may be expressed by saying that the yellow fringes are more closely crowded together than the red.

If the position of the two systems of fringes is marked on a

straight line Ox (supposing that the point O corresponds to $\Delta = 0$, and that the values of Δ increase along Ox in proportion to the distance from the point O), a figure like the following will result (Fig. 3):¹ the two systems of fringes, at first confused,

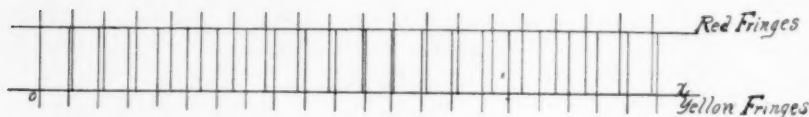


FIG. 3.

separate little by little; then a yellow fringe falls about half way between two red ones; again there is approximate coincidence of the two systems of fringes, or, more correctly, two consecutive red fringes are found, which comprise not one only, but two yellow fringes. Further on separation again occurs, then a new coincidence, etc. Let us find when this phenomenon of coincidence is produced. Let A and A_1 (Fig. 4)² be two consecutive red fringes, of order K and $K + 1$, between which are found two

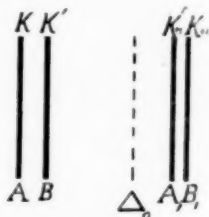


FIG. 4.

yellow fringes B and B_1 of order K' and $K' + 1$, and let $K' - K = m$. In B , p' is equal to K' , while p is a little greater than K ; we therefore have $p' - p < K' - K$, or $p' - p < m$; similarly in B_1 , $p' - p > m$. As, moreover, $p' - p$ increases in proportion to Δ , we find between A and B a certain value Δ_0 of the difference of path for which

$$p' - p = m. \quad (1)$$

¹ To distinguish the two systems of fringes the red have been prolonged toward the top and the yellow toward the bottom.

² In Fig. 4, interchange A, B , to read B, A .

The difference of path thus defined will be called the difference of path of coincidence; it is defined by equation (1), which may be written

$$\frac{\Delta_o}{\lambda'} - \frac{\Delta_o}{\lambda} = m, \text{ or } \Delta_o = m \frac{\lambda \lambda'}{\lambda - \lambda'}.$$

It is evident that the phenomenon is periodic, repeating itself for values of Δ which are multiples of a length

$$\Pi = \frac{\lambda \lambda'}{\lambda - \lambda'},$$

which we will call the *period*. This quantity may be expressed in wave-lengths from any radiating source; expressed as a function of λ , it will have the value

$$\omega = \frac{\lambda'}{\lambda - \lambda'},$$

and as a function of λ'

$$\omega' = \frac{\lambda}{\lambda - \lambda'} = \omega + 1.$$

The m^{th} coincidence occurs when the order of interference corresponding to the yellow radiation exceeds by the integral number m that which corresponds to the red radiation; it is repeated for the difference of path having the value $\Delta = m \Pi$; $m \omega$ and $m \omega'$ are the corresponding values of the order of interference. If $m \omega$ is an integral number K there will be *exact coincidence* between the K^{th} red fringe and the $(K + m)^{\text{th}}$ yellow fringe. In the general case, $m \omega$ will be a fraction $K + \theta$ ($0 < \theta < 1$), and the appearance will be that shown in Fig. 4. The fraction θ characterizes the appearance of this *approximate coincidence*, for, if we designate by a and a_1 the distances AB and B_1A_1 , we have

$$\theta = \frac{a}{a + a_1}.$$

Observation furnishes a check on the value of θ , at least when the two radiations employed differ sufficiently in wave-length.

Discordances are similarly defined by the condition that the

difference $p' - p$ shall be equal to an integral number plus $\frac{1}{2}$; the values of Δ corresponding to the discordances are

$$\Delta = (m + \frac{1}{2}) \Pi.$$

They are recognized by the fact that a fringe of one kind occurs practically in the center of the dark interval which separates two fringes of the other kind

It is evident that all of these phenomena depend on the *period*, which may be calculated when the two wave-lengths are known. If the two radiations are far apart in the spectrum, as for instance, one in the green and one in the red, the period is short (for example, 4 or 5 fringes); the separation of the red fringes is sensibly greater than that of the green; the coincidences are repeated at short intervals, and they are recognized the more easily as the colors of the two systems of rings differ more widely, reducing the possibility of any confusion between them. It is even easy to distinguish the exact from the inexact coincidences, and observation gives an indication of the value of θ which establishes the inexactness of the coincidence.

If, on the contrary, the two radiations differ but little in wave-length, as, for example, the two sodium lines, the period is long (about 1000 in this case); the position of coincidence can be determined by observation only within a few fringes; the two kinds of rings have almost exactly the same color, and their separation, which gives rise to no lack of symmetry in color, becomes appreciable only a certain number of fringes before or after coincidence, by the widening of the rings.

Such are the phenomena which arise from the superposition of two systems of fringes. We have made no hypothesis regarding the method of observing them; the phenomena will remain precisely the same whether it is a question of *fringes of thin films* observed in parallel light, or whether the fringes produced by a layer of air of uniform thickness are observed in convergent light.

III. APPLICATION TO SPECTROSCOPY.

It follows from what has been stated that the observation of a system of fringes from silvered plates furnishes a means of

separating two radiations of different wave-length, and consequently of making a *spectroscopic study* of a mixture of radiations. It is easy to see that the resolving power of the apparatus increases with the order of the fringes observed. Consider two radiations of nearly equal wave-length, λ and $\lambda' = \lambda + \epsilon$. Under what conditions can they be separated? Experience shows that the separation of the two systems of rings is always clearly visible when the distance between rings of different kinds is $\frac{1}{5}$ the distance between consecutive rings of the same kind. It will suffice, to effect separation, to reach the fringe whose order is $\frac{1}{5}$ the period, or of the order $p = \frac{1}{5} \frac{\lambda}{\epsilon}$, corresponding to a distance between the silvered surfaces $e = p \frac{\lambda}{2} = \frac{\lambda}{10} - \frac{\lambda}{\epsilon}$. Suppose, for example, that a distance between the silvered surfaces $e = 5\text{ cm}$ has been reached; the corresponding order will be, supposing $\lambda = 0.5\mu$, $p = 200,000$, and two radiations such that $\frac{\epsilon}{\lambda} = \frac{1}{1000000}$, where the distance apart in the spectrum is only $\frac{1}{10000}$ of the distance between the D lines, can be separated. Beyond the distance $e = 5\text{ mm}$ ($p = 20,000$) it is possible to resolve two radiations whose distance apart is less than $\frac{1}{100}$ that of the D lines; *i. e.*, the power of the apparatus is already comparable with that of the best spectroscopes having prisms or gratings.

In order to effect such a result, it is necessary to be able to obtain very sharp fringes with very great differences of path; the employment of fringes in convergent light is thus indicated. We use the apparatus permitting a parallel displacement, which has already been described. Suppose that it is desired to study with this apparatus an approximately monochromatic light. It is illuminated with this light, and the distance between the two silvered surfaces is gradually increased. If the radiation under examination is multiple, each ring will be seen to separate successively into several others; each of these partial rings corresponds to a monochromatic radiation, and the more refrangible radiations are on the inside. Each fringe constitutes a veritable spectrum of the luminous source; the apparatus is thus similar to

a grating, the resolving power of which is certainly small, but with which it is possible to observe spectra of very high order, which permits its resolving power to be increased almost indefinitely.

When the light is complex it is easy to obtain a precise measure of the ratio of the wave-lengths of the radiations which constitute it; let there be two radiations of nearly equal wave-length, λ and $\lambda + \epsilon$. The distance between the silvered surfaces is increased until the discordance between the two systems of rings is complete. Then, if e is the distance between the surfaces (which is given with sufficient accuracy by the micrometer),

we have $\frac{\epsilon}{\lambda} = \frac{\lambda}{4e}$.

We have studied in this way a certain number of radiations emitted by metallic vapors illuminated by an induction discharge (mercury, cadmium, thallium).¹ Our results are not identical with those deduced by Professor Michelson from his investigations on the visibility of fringes; but it should not be forgotten that Professor Michelson's method does not permit the complete determination of the constitution of a group of lines; an infinite number of hypotheses on the constitution of the group can be made to correspond to a single visibility curve, and the result is therefore in large degree arbitrary. In fact, our results would lead to visibility curves identical with those found by Professor Michelson; far from contradicting the experimental results of this investigator, our researches confirm them completely.

This method is also readily adapted to the study of the change of wave-length of a given line, on condition that the radiation be sufficiently near monochromatic; in such a case a comparison can be made of two sources emitting, for instance, in the one case the altered radiation, and in the other the normal radiation, attention being directed to the change in the appearance of the rings produced by the two sources successively.

¹In certain cases, it is necessary to separate out radiations which would be troublesome on account of a complication of colors, or even, if the wave-lengths differ but little, because of the confusion of the rings; we have employed for this purpose either tanks of absorbing liquids, or one or more carbon bisulphide prisms, used in the ordinary way.

IV. DETERMINATION OF THE ORDER OF A FRINGE.

This problem presents itself in all determinations of length by interference methods. It is not ordinarily practicable to count the fringes beginning with which corresponds to zero difference of path, either because the zero fringe is not accessible, or simply on account of the difficulty of counting a number of fringes which may attain hundreds of thousands, if a measurement of a length of several centimeters is involved.

Our method is based on the observation of coincidences of fringes produced when the incident light contains two monochromatic radiations. It has been seen that this phenomenon is periodic, so that there exist certain fringes, of which a list may be made, which are, so to speak, characterized by the same distinctive sign. If the two radiations differ greatly, the fringes thus characterized are distinguishable without difficulty, but they are numerous and make a long list. On the contrary, when the two radiations differ but little, the fringes characterized by a distinctive mark succeed one another at long intervals; they are few in number and it is not difficult, when one is seen, to find it in the table. But observation will not suffice to designate the particular fringe with precision; it can only be said that it occurs in a certain region. By a careful choice of radiations combined in pairs, it is possible to determine the exact number of a fringe, provided its roughly approximate value is already known.

The method of coincidences is applicable whatever mode of observing the fringes be employed. In the case of the lower orders, they can be observed in parallel light, in the form of fringes from thin films. It then suffices to have the radiations employed approximately monochromatic; those given by alkaline salts in the flame serve perfectly. Thus in the verification of the order of the fringes furnished by our *standard films* (see below), we have employed the radiations of sodium and lithium in the flame of a Bunsen burner or an oxy-hydrogen blowpipe.

If it is desired to pass to fringes of higher orders, interference is produced in *convergent light*, and it is necessary to have

recourse to the monochromatic radiations from the induction discharge in a metallic vapor. We have employed the brightest and most nearly monochromatic lines available in order to render possible the production and enumeration of fringes of a high order. These are the red and green lines of cadmium, the two yellow lines and the green line of mercury. To produce them the two tubes containing metallic vapors (Michelson tubes) are placed in series in the secondary of an induction coil. They occupy the foci of two convex lenses whose axes meet in a right angle. At the point of intersection is placed a lightly silvered glass plate, or a pile of plates which is traversed by one of the beams, while the other is reflected; we thus obtain the complete superposition of two beams, as though they came from the same light-source. Two movable screens, which the observer controls by means of cords without moving from his seat, permit the light from either tube to be cut off. Small tanks of colored liquids, which are placed directly before the eye, are used to cut out superfluous radiations.

In what follows we will refer everything to the fringes given by the green light of cadmium; the periods of coincidence will be expressed in terms of the wave-length of this radiation.

We group the radiations as follows:

1. The two yellow lines of mercury

$$\lambda = 0.57906593 \mu, \quad \lambda = 0.57695984 \mu.$$

These two lines are close together in the spectrum (about three times the distance of the D lines); their period of coincidence is 311.9 (expressed in terms of the wave-length of the green cadmium line). We observe the coincidences, which can be determined within about twenty fringes, *i. e.*, in the twenty fringes which precede or follow the coincidence the separation of the two systems of rings is not appreciable.

2. The green line of cadmium ($\lambda = 0.50858240 \mu$) and the green line of mercury ($\lambda = 0.54607424 \mu$) have a period of 14.56515. We observe the discordances, and the observation determines without question the fringe for which this phenomenon is produced.

The green and red ($\lambda = 0.64384722 \mu$) radiations of cadmium, which differ widely, have a period of 4.759901. We observe the coincidences, and this observation is greatly facilitated by the great difference in color of the two systems of rings. Observation gives to within about 0.1 the fraction which denotes the exactness of the discordance.

We now come to the determination of the order of a fringe.

The divided scale attached to the carriage of the interference apparatus, gives the distance between the silvered surfaces within a few hundredths of a millimeter. This measurement suffices to determine between what coincidences of the two yellow lines the observed fringes lie. Moreover, observation of these coincidences themselves gives a determination of the reading which corresponds to zero distance, and calibrates the scale with a sufficient degree of precision.

This being understood, as the coincidences of the red and green radiations of cadmium occur at short intervals, one of them is always in the field of the telescope; let us consider one of the green rings which encloses this coincidence, and let K be its order, which it is desired to determine.

Cutting out the red radiation, we superpose the green radiations of cadmium and mercury; then, slowly varying the distance between the silvered plates by means of the arrangement for producing flexure, we count the number of cadmium fringes, starting from fringe K , which must be caused to pass in order to produce discordance between the two systems of green rings. Call C this number, which cannot exceed fourteen, because coincidence of the two green lines occurs every fourteen fringes, and which is even less than seven, if care is taken to produce the displacement in the most favorable direction.

We continue to change the distance until a coincidence of the two yellow lines of mercury is reached, and during this motion we count the number C' of coincidences of the two green lines which pass across the field; the motion need not be very slow, since we no longer count the *fringes*, but the *coincidences*, which are fourteen times less numerous. The number of green

fringes of cadmium which have passed during this motion is about $C' \times 14.57$.

Finally, let m be the number of the coincidence of the two yellow lines which has been reached, a number known from the approximate measurement of the thickness by means of the divided scale. It suffices to know the three integral numbers C , C' , m , in order to solve the problem.

Suppose, to make the matter clear, that the two motions thus effected have resulted in bringing the plates nearer together. The m^{th} coincidence of the two yellow lines occurs when the number of the green cadmium fringe is $311.9 \times m$. To go from this to the discordance of the two green lines which was observed near K , $14.57 \times C'$ fringes had to pass; as the coincidence of two yellow fringes is observed to only about ± 20 fringes, the number of the discordance fringe of the two green lines will be

$$311.9 \times m + 14.57 \times C' \pm 20.$$

The number of discordance fringes in this interval is next calculated; there will be three or four at most, among them the one which has been observed. Further, on adding the number C to the number of this discordance fringe, we should encounter a coincidence of the green and red lines of cadmium; having calculated a table of coincidences, a choice will be made without hesitation. An important check will be given by the fact that observation gives with a precision of 0.1 the fraction which denotes the inexactness of coincidence of the two cadmium lines; the observed fraction should agree with its calculated value.

It is evident that the direct result of observation is reduced to three integral numbers, one of which is given directly by reading a divided scale, the two others being each less than 15. This method is applicable up to thicknesses of 4 cm or 5 cm, and consequently renders possible the rigorous determination of numbers of fringes which may reach as high as 200,000, by counting only the coincidences, the numbers to be counted being less than 10, and the quickly-obtained result being easily found

anew for the purpose of verification. The application of the method requires the use of no special measuring instrument, such as a micrometer, compensator, etc.; it is sufficient to be able to observe the few fringes which lie near those which it is desired to study.

V. COMPARISON OF WAVE-LENGTHS.

The method just described requires that the ratios of the wave-lengths of the radiations employed be exactly known. A precision of $\frac{1}{1000000}$ is not sufficient.

For the lines of cadmium, the ratios of the wave-lengths are known with a precision which leaves nothing to be desired, thanks to the beautiful investigations of Professor Michelson. The same is not true of the mercury lines, which are known only from old measures made with gratings, the precision of which is far from sufficient. We have, therefore, compared the wave-lengths of these radiations with those of cadmium, by the observation of interference phenomena. This comparison can be made by means of observations identical with those which serve for the determination of the order of fringes, provided that fringes of a low order are used at the outset, and subsequently those of higher and higher orders.

The old measures give a first approximation of the ratios sought, with which the approximate values of the periods of coincidence of these radiations among themselves, or with the cadmium lines, can be calculated. It is consequently possible to calculate approximate tables of the coincidences, and these tables will contain only small errors, such that the order number of the fringes will reach only a moderate value (*e. g.*, a few thousands); moreover, the coincidences of the cadmium fringes among themselves can be exactly calculated by using the values obtained by Professor Michelson.

Setting the two surfaces of the interference apparatus a short distance apart (1 mm for example), the observations of coincidences are made just as though it were merely a question of determining the order of a fringe. In comparing the observa-

tions with the approximate tables of coincidences, it will be found that only a single hypothesis regarding the order of the observed fringes will permit the observed phenomena to be brought into close accordance with the results of calculation. If the accurate identification is not readily made it is because the errors of the tables of coincidences are still too great for the orders of the observed fringes, and it will be necessary to repeat the observation on fringes of lower orders. The more inexact the values used in the preliminary calculations, the lower the order of fringes that must be chosen to avoid all doubt.

Only one hypothesis being admissible, the small discrepancy remaining between observation and calculation indicates that the values used for the wave-lengths are not quite exact, and permits them to be corrected.

These new values render possible the calculation of more exact tables of coincidences, by means of which the same operations can be repeated with fringes of a higher order, double the previous one, for example. This new observation furnishes the means of again correcting the wave-length tables, and continuing thus, we obtain more and more precise values as fringes of higher orders are observed.

In brief, the experiment consists in observing the reciprocal position of two systems of rings, using for this purpose coincidences or discordances; an observation of this character fixes the reciprocal position of two fringes within at least $\frac{1}{20}$ of a fringe. In comparing two wave-lengths, λ and λ' , the relative error possible for the second, supposing the first known, will be, if the observations are discontinued at fringes of order p ,

$$\frac{d\lambda'}{\lambda'} = \pm \frac{1}{20p}.$$

A very high degree of precision is soon attained; for example, let $p = 10,000$, which corresponds to a distance of less than 3 mm between the silvered surfaces, the possible relative error will be $\frac{1}{200,000}$, a precision which it would doubtless be difficult to surpass with gratings.¹

¹ This is particularly true of lines widely separated in the spectrum, since in this case the errors of the divided circles will enter.

Our measures have been carried to a separation of the silvered surfaces amounting to 32 mm (order about 125,000 for the green cadmium fringes). The observations once completed, it is desirable to utilize all of them for the definitive calculation, applying the method of least squares to the series of equations which they furnish.

The values below, based on Professor Michelson's values of the wave-lengths of the cadmium lines, are referred to the latter in *air* at 15° , under a pressure of 760 mm; the ratios of these numbers remain sensibly constant under ordinary atmospheric conditions. The probable error is 5 units in the last place, or $\frac{1}{10000000}$ in relative value.

It is evident that interference methods permit wave-lengths to be compared with a remarkable degree of precision. These methods have the advantage of being based directly on the *definition* of the quantity measured, and of being free from all systematic error due to delicate instruments (gratings, divided circles, etc.), which are encountered in other methods.

We have thus found

$$\begin{aligned} \text{Yellow lines of mercury } & \left\{ \begin{array}{l} \lambda_1 = 0.57906593 \mu \\ \lambda_2 = 0.57695984 \mu \end{array} \right. \\ \text{Green line of mercury } & \lambda = 0.54607424 \mu \end{aligned}$$

VI. MEASUREMENT OF LENGTHS.

The determination of the order of a fringe makes known the whole number of wave-lengths contained in a given length. In order to have a measure expressed in wave-lengths, it only remains to determine a fraction, which does not need to be known with a very great relative precision on account of the extreme minuteness of the wave-length.

Let it be required to determine the thickness, at a given point, of a *thin film* of air between silvered surfaces. By illuminating the system with a parallel beam of monochromatic light, a system of fringes will be obtained; the position of the given point with reference to the two fringes which encircle it is determined, and finally the order of one of these fringes is sought.

In the case of a film with parallel faces giving rings at infinity, the order K of the ring immediately surrounding the center will be found. The order of interference corresponding to the center of the system is a little greater than K , say

$$K + \eta, \quad (0 < \eta < 1);$$

and the distance between the two silvered surfaces will be $(K + \eta) \frac{\lambda}{2}$; this will be known if the fraction η is determined.

The simplest means of doing this is to find the angular diameter of the ring K , using for this purpose, for example, an eyepiece micrometer. Call this diameter $2i$. We have $K = \frac{2e \cos i}{\lambda}$,

whence $e = K \frac{\lambda}{2 \cos i}$. The fraction η , which it is really unnecessary to calculate, would have the value

$$\eta = K \frac{1 - \cos i}{\cos i} = K \frac{i^2}{2},$$

remembering that i is very small.

The use of these methods of measuring thicknesses presupposes the possibility of applying the method already described for the determination of the order of a fringe; for this a certain number of conditions must be met: among others, it is necessary to be able to examine a certain number of fringes in the neighborhood of the one whose order is required. This condition greatly limits the cases in which these measuring processes can be applied. An optical method by which it would be possible to establish the equality of two thicknesses would evidently possess great interest; it would thus be possible to compare the length to be measured with a thickness measured beforehand, or also to copy a given length by means of a system which is kept under conditions favorable to purposes of measurement. We have succeeded in solving this problem, thanks to the use of fringes in white light, the theory of which we will give. Our method rendering it possible to double, triple, etc., a thickness will permit the measurement of lengths much greater than those which can be determined directly by interference methods.

Superposition fringes.—These fringes are produced when a beam of white light traverses successively two air films bounded by silvered surfaces, A and A' , of suitable thicknesses. An incident ray can, in fact, give rise to two emergent rays, one of which has passed directly through A and has been twice reflected in A' , while the other has passed directly through A' and has been twice reflected on the surfaces of A . The difference of path of these two rays is $\Delta - \Delta'$, if Δ and Δ' are the differences of path corresponding to each of the films for the ray considered. It is zero if $\Delta = \Delta'$. We will thus have, if the two differences of path are nearly alike, and consequently the thicknesses of the two films not far from equal, a system of fringes in which the central white fringe marks out the position of points such that $\Delta = \Delta'$. This central fringe is bordered by brilliant colors.

Fringes in white light appear not only when the two thicknesses are nearly equal, but also when they stand in a simple ratio; they are then due to the interference of rays which have undergone an unequal number of reflections. The systems of fringes corresponding to $\frac{\Delta}{\Delta'} = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{3}{2}, \frac{4}{3} \dots$ can be easily observed.

However, in proportion as this ratio becomes less simple the fringes are fainter, since they are due to the interference of rays which have undergone a greater and greater number of reflections, and since an increasingly important fraction of the light cannot interfere and produces white light, which diminishes the contrast of the fringes.

As for the manner of observing these phenomena, this can be varied according to circumstances. In the case of small thicknesses they are observed in parallel rays; with thick layers and uniform thicknesses, the fringes will be observed at infinity in convergent light.

Superposition fringes in parallel light; measurement of small thicknesses.—The beam of white light in this case passes normally through the two thin films A and A' , which will be placed

directly in line with one another, or better *superposed optically*, by projecting on the second, by means of an optical system, an image of the first. The fringes are then localized in this plane, which contains at once the second film and the image of the first. At one point in this plane the thicknesses e and e' correspond for the two films; the corresponding differences of path are $\Delta = 2e$, $\Delta' = 2e'$. If in a certain region $\frac{e}{e'}$ approximates the commensurable ratio $\frac{p}{q}$ a system of fringes in which the central fringe is defined by $\frac{e}{e'} = \frac{p}{q}$ is obtained. Or again, if in a certain region the two thicknesses differ but little, we obtain a system of fringes in which the central fringe outlines the region of points such that $e = e'$.

From this we have a means of establishing the equality in thickness of two thin films at given points: the image of one is projected upon the other, so that the two given points correspond. If the thickness at these points is equal they must occur on the central fringe.

On this consideration we have based the construction and use of *standard films* for the almost instantaneous measurement of small thicknesses; it is clear, in fact, that if the film A' has been calibrated, *i. e.*, if its thickness has been determined at different points, the point in this film where the thickness is equal to that which it is wished to measure may be sought. The measurement is extremely rapid, the standard plate taking the place of a divided scale, the graduation of which can be controlled, when desired, by means of fringes in monochromatic light.

We will also point out, as an application of these superposition fringes, the solution of the following problem which may arise in the construction of various measuring instruments. Two parts of an apparatus are susceptible of small displacements with respect to one another; it is desired to supply the system with a reference mark so that it can be brought back to the same relative position. Ordinarily a microscope attached to

one of the parts, and focused on an index carried by the other, is employed for this purpose. The use of superposition fringes renders it possible to fix the reference position within a few thousandths of a micron.¹

Superposition fringes in convergent light; measurement of great thicknesses.—Great difficulties will be encountered if it is desired to obtain by the preceding methods superposition fringes with moderately large thicknesses. It is then advisable to employ plates of uniform thickness and to observe the fringes in convergent light with a telescope focused for parallel rays.

Let L and L' be the two plates (Fig. 5); both of them have parallel faces, but the faces of the first make a small angle

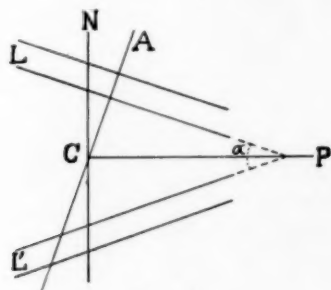


FIG. 5.

α with those of the second. The plane of the figure is normal to the two systems of faces; their bisecting plane is directed along CP , and the normal to this plane is CN . Suppose at the outset that the thicknesses e and e' of the two plates are nearly equal; there may then be interference between the wave which has passed directly through L and has been twice reflected on the faces of L' with that which has pursued the reverse course. Consider a direction making a small angle with CN , and which is projected on the plane of the figure in CA . To this direction

¹ See *Annales de Chimie et de Physique*.

An effective application of this method has been made by us. ("Sur un nouveau voltmètre électrostatique interférentiel" *Jour. de Phys.*, November 1898.)

corresponds a single value of the difference of path between the two interfering waves, the value of which is

$$\Delta = 2e \cos i - 2e' \cos i',$$

i and i' being the angles of incidence of the given direction in the two films. Remembering that i and i' are very small, and that e' differs but very little from e , this expression may be written

$$\Delta = 2(e - e') + e(i'^2 - i^2),$$

or by a simple transformation

$$\Delta = 2(e - e') + 2ea\theta,$$

θ being the angle made by CN with the projection CA of the direction considered.

It is seen that Δ varies proportionally with θ . We will thus have in a telescope focused for parallel rays a system of rectilinear fringes, equidistant and perpendicular to the plane of the figure, *i. e.*, parallel to the intersection of the faces of the two plates. The angular distance of two consecutive fringes is $\frac{\lambda}{2ea}$, and the central fringe is defined by $\theta = \frac{e - e'}{ea}$.

The entire system of fringes is displaced parallel to itself if one of the thicknesses is varied; the central fringe moves toward the point in the field which corresponds to the direction CN ($\theta = 0$) when $e = e'$. Moreover, the fringes broaden if the angle a is diminished, and for $a = 0$ Δ approaches the constant value $2(e - e')$; the system of fringes tends to acquire a uniform color, which is white if $e = e'$.

From this we derive a method of establishing with a high degree of precision, the equality of two thicknesses each corresponding to the distance between two plane parallel surfaces of silvered glass. If there is a very small difference between the two thicknesses, it can be measured with precision; finally if one of the thicknesses is susceptible of slight variation, it can be brought into exact equality with the other, and thus a given thickness can be *copied*.

Phenomena entirely similar to those just mentioned will be obtained if the thickness e' , instead of being equal to e , is, for example, one half or one third of it. From this is derived a

means for exactly multiplying a given length by $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, . . . or by 2, 3, 4, . . . As we are dealing here with phenomena in *white light*, we can apply these methods even to very great thicknesses, and although the adjustments become more difficult in proportion as the distances increase, it seems possible, under good conditions, to observe superposition fringes with thicknesses of at least 1 m.

Moreover, the distance between two parallel silvered surfaces can be measured directly in wave-lengths, provided that it does not exceed 4 cm or 5 cm and is susceptible of slight variation. Combining this method of measuring with the use of superposition fringes, it becomes possible to measure the given and invariable distance between two parallel silvered surfaces, even if it greatly exceeds the limit just given.

Let it be required to measure the thickness of the layer of air L (Fig. 5) which we will suppose at first to be less than 5 cm. Employing the superposition fringes, this thickness is copied by means of the film L' , the thickness of which can be varied at will; then this latter is measured by the methods indicated. It should be remarked that the operation can be, so to speak, instantaneous, and gives the measure of the desired thickness at a given instant, without requiring that this distance be invariable.

If the thickness of L exceeds 5 cm the same method is followed, but instead of copying it, the half, third or quarter is taken, which will render possible the measurement in wave-lengths of thicknesses up to 20 cm in a single operation.

Finally, for greater thicknesses, the same procedure may be followed, by taking a certain number of intermediate standards. Let it be required to measure the thickness of the layer L_1 , too great to permit the preceding method to be applied. By means of a layer L_2 of variable thickness, we can, for instance, take a quarter of it, then a quarter of this latter by means of the layer L_3 , and so on, until a layer L_n of thickness less than 5 cm, directly measurable in wave-lengths, is reached. By means of easily devised arrangements the whole of these operations can

be rendered almost instantaneous, thus removing all danger of error arising from any change occurring during measurement.

All the lengths to be measured by these methods must be represented by the distance between two plane parallel silvered surfaces placed facing each other. But it is easy to pass from this case to that where it is required to measure the thickness of a solid with polished and sensibly plane parallel surfaces, like the thickness of a parallelepiped of glass. It suffices to place this solid between the parallel silvered faces of a suitable system, which will play the part of *calipers*; the distance of the surfaces is adjusted in such a manner that there remain only very small thicknesses of air between them and the faces of the solid. The use of our *standard films* (see above) permits this last measurement to be made rapidly.

We have applied these processes to the measurement in wave-lengths of the thickness of a glass cube 3 cm on an edge. The measurement was made with precision, in spite of the very defective conditions under which it was effected; almost the entire apparatus was constructed of wood; on account of the vibration of the soil it had to be suspended by means of rubber rings; no precaution was taken to avoid temperature variations. The success of the experiment under such unfavorable conditions was evidently due to the fact that the measurement was instantaneous, and consequently free from any error arising from a modification of the apparatus.

The same methods would evidently be applicable to much greater thicknesses. We may hope to be able to measure in wave-lengths, with no microscope settings, the length of a "mètre à bouts." The conditions under which our experiments have been made did not permit us to attempt so delicate an application of the method with any chance of success; our purpose was only to render evident the possibility of such a measurement.

Such are, in brief, the principal applications that we have made of interference phenomena given by silvered plates. The greater part of the problems that we have attacked have already

received other solutions ; our methods are notable for the simplicity of the apparatus, the work of the constructor being reduced to the figuring of two plane surfaces. This is evidently a good means of avoiding systematic errors, and of utilizing as completely as possible the remarkable power of interference methods. Particularly for the measurement of lengths and all allied problems, these methods can render effective service with the ordinary resources of a laboratory. The difficulties resulting from the extreme minuteness of the wave-length are largely offset by the precision of the measures and the certainty offered by the standard employed.

MINOR CONTRIBUTIONS AND NOTES.

ADDITIONAL OBSERVATIONS OF EROS (433).¹

THE method of search for Eros (433), described in *Circular No. 36*, has been continued. The ephemeris has been extended by Mr. Chandler, as required, and images of the planet have been found by the writer on thirteen plates. From these images the following approximate positions have been determined in addition to those given in *Circular No. 36*. The successive columns give the number of the plate, the date, the Greenwich mean time of the middle of the exposure, the length of the exposure in minutes, and the approximate right ascension and declination for 1875. These positions are, in general, derived from adjacent Durchmusterung stars. Preparations are now being made for precise determinations of the positions of the planet on these plates, and on those described in *Circular No. 36*. The last two columns give the anomaly and the computed photographic magnitude, assuming the magnitude at distance unity, 12.0, as derived from the measures given in *Circular No. 34*. The correction for phase is necessarily omitted, and may exceed a magnitude, as the phase angle may amount to 60°. The last three plates were taken at Arequipa, all of the others at Cambridge.

Plate	Date			G. M. T.		Ex.	R. A. 1875		Dec. 1895		v.	Mag.
	y	m	d	h	m	m	h	m	°	'	°	
I 9801	1893	10	28	21	55	14	5	58.8	+ 53	40	— 71	10.9
I 9832	1893	10	30	20	18	10	6	4.5	+ 54	6	— 69	10.8
I 9862	1893	10	31	21	21	15	6	7.6	+ 54	20	— 69	10.8
I 10095	1893	11	26	20	26	17	7	17.5	+ 57	50	— 49	9.9
I 10280	1893	12	23	19	49	13	7	45.7	+ 52	58	— 26	8.8
I 10407	1894	1	8	18	8	65	7	35.5	+ 41	23	— 12	8.4
I 10469	1894	1	19	16	57	10	7	26.5	+ 28	46	— 3	8.2
I 10559	1894	1	25	16	16	13	7	23.6	+ 21	15	+ 3	8.2
I 10604	1894	1	30	13	40	60	7	22.5	+ 15	20	+ 7	8.3
A 222	1894	2	5	15	26	60	7	23.3	+ 8	46	+ 12	8.4
B 11174	1894	5	19	14	16	10	10	38.0	— 14	57	+ 92	11.9
B 16518	1896	6	29	19	17	15	17	37.8	— 36	22	+ 152	12.5
A 1876	1896	6	30	13	46	60	17	36.4	— 36	12	+ 152	12.5

¹ *Harvard College Observatory Circular No. 37.*

I 9801. This photograph is important, since, with that taken on May 19, 1894, the anomaly through which the planet was observed in 1893-4 becomes 162° . The observations contained in *Circular* No. 36 extended over an angle of 101° .

I 10280. Plate dark, and difficult to measure.

I 10407. Faint spectrum on edge of plate.

I 10469. This plate was fogged and was so dark that it was marked useless. Its density was about that of a shade glass used in viewing the Sun. By making a double contact print from it a photograph is obtained on which accurate measurements of the planet are possible.

I 10604. Spectrum. Superposed on spectrum of $+15^\circ$ 1581, Mag. 9.5.

A 222. Well marked trail 160" long, showing irregularities in running of driving clock of telescope.

B 11174. See I 9801.

A 1876. Well marked trail. At about $13^h 15^m$ the planet would have coincided very nearly with -36° 11846. The orbit of the planet could be well determined from the observations in 1896 alone, using for the first place the position of April 6, for the second the three positions on June 4 and June 5, given in *Circular* No. 36, and for the third, this photograph with that taken on June 29.

Some important conclusions may be derived from this investigation. All the photographs on which the planet has been found were taken with doublets. If they had been taken with lenses of the usual form with a field 2° in diameter all of the images would have fallen outside of the plates. In view of the difficulties found in photographing this object with an ordinary lens at Greenwich and Oxford (*Observatory* 21, 429) it is doubtful whether we should have obtained many images of it here with such a lens, even if it had been in regions photographed. The number of plates on which the planet appears probably fairly represents the number we have of all other similar objects, whether already discovered or not. This planet is bright during only a small portion of time. During the last eleven years it has been brighter than the ninth magnitude, photographically, for only two months, or about a seventieth part of the entire time. There may be other similar objects, even brighter, as yet undiscovered. Nova Aurigae was as bright as the fifth magnitude for six weeks before it was discovered. Had Eros attained the sixth magnitude instead

of the eighth it should have appeared on plates taken with the transit photometer. In this case, we should have had an image of it on every clear night on which it culminated after dark. Fairly good positions could have been obtained from these images since the focal length of the telescope is about two feet, and the exposures are so short that the images are always circular. We have now a similar instrument in Arequipa, so that, in general, two images should be obtained every night.

EDWARD C. PICKERING.

JANUARY 16, 1899.

AN ANNUAL REPORT ON THE PROGRESS OF
ASTRONOMY.

I INTEND to publish an *Astronomischer Jahresbericht mit Unterstützung der Astronomischen Gesellschaft* (Astronomical Yearly Report aided by the Astronomische Gesellschaft). It will give short reports of all the works on astronomy, astrophysics and geodesy, both practical and theoretical which have appeared during the year. The first volume will appear in 1900 and will contain reports of all the publications of 1899. Not wishing to overlook anything I should be much obliged if all authors of such publications, appearing as separate books or articles in journals not usually destined and used for astronomical publications, would kindly communicate them to me.

WALTER F. WISLICENUS.

NICOLAUSRING 37
STRASSBURG (ELSASS)
January 1899.

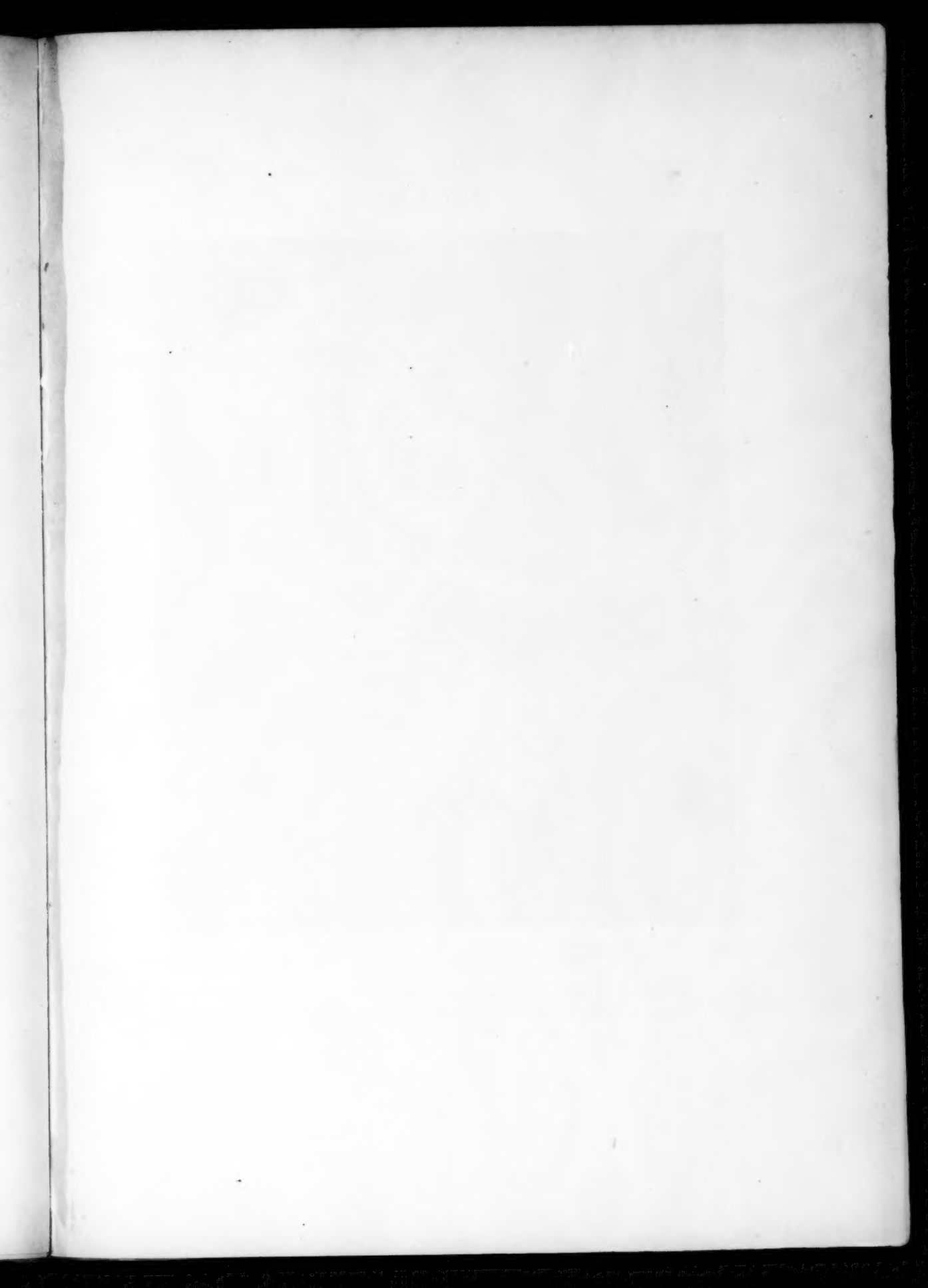


PLATE II.



PHOTOGRAPH OF THE MILKY WAY NEAR THE STAR THETA OPHIUCHI.